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A NOTE ON ESTIMATING PARAMETERS OF SOME SPECIAL TIME SERIES

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Abstract. *Some possibilities of application of the composition of methods of moments and least squares for estimating parameters of some special types of the first order autoregressive time series with exponential marginal distribution are discussed in this paper.*

1. Preliminaries

Many of the first order autoregressive time series with exponential $\mathcal{E}(\lambda)$ marginals can be represented by the first order stochastic difference equation

$$X_t = U_t X_{t-1} + V_t, \qquad t = 0, \pm 1, \pm 2, \cdots \tag{1.1}$$

defined by Popović [6] in the following way:

I The sequence of random variables $\{X_t\}$ is semi-independent of the random sequences $\{U_t\}, \{V_t\}$ and $\{E_t\}$ (X_i and U_j, or V_j, or E_j are independent if and only if $i < j$).

II $\{E_t\}$ is the sequence of independent identicaly distributed (i.i.d.) random variables with exponential of $\lambda, \mathcal{E}(\lambda)$ distribution and E_t is independent of U_i and V_j for every t, i, j.

III $\{U_t\}, \{V_t\}$ and $\{(U_t, V_t)\}$ are i.i.d. sequences of discret random variables and vectors which satisfy the following special conditions:

(a) $P(0 \le U_t \le 1) = 1$
(b) $P(0 \le V_t \le 1) = 1$
(c) $0 < E(U_t^2), E(U_t) < 1$
(d) $E(V_t) = 1 - E(U_t)$
(e) $E(U_t^2) + E(V_t^2) = 1 - E(U_t V_t)$
(f) If $P(U_t = \alpha_i) = p_{U_i}$, $i = 1, \cdots, k$, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k \le 1$, $\sum_{i=1}^{k} p_{U_i} = 1$

and $P(V_t = \beta_j) = p_{V_j}$, $j = 1, \cdots, r$, $0 \le \beta_1 < \beta_2 < \cdots < \beta_r \le 1$, $\sum_{j=1}^{r} p_{V_j} = 1$ then

$\sum_{i=1}^{k}\sum_{j=1}^{r} \frac{p_{U_i} p_{V_j}}{\alpha_i - \beta_j}\left(\frac{\alpha_i}{\lambda + s\alpha_i} - \frac{\beta_j}{\lambda + s\beta_j}\right) = \frac{1}{\lambda + s}$ for any real s.

In the same time, it was proved that the equation (1.1), under the above mentioned conditions, has unique, stationary, strong stationary and ergodic solution.

This model differs from the RCA models defined by Nicholls and Quinn [5] in the fact that those models had independent random coefficients.

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If we assign for convenience:

$$
(1.2)\qquad \begin{array}{ll} E(U_t)=a & E(V_t^2)=c \\ E(U_t^2)=b & E(X_t)=\dfrac{1}{\lambda}=m, \end{array}
$$

we shall use the estimates of a, b, and c supposing that λ is known to discover the estimates of the parameters of some special mixtures of distributions.

The method of least squares in two steps has been applied in [7] to estimate moments a, b and c of dependent random variables U_t and V_t supposing that m is known. These estimates are

$$
(1.3)\qquad \begin{aligned}
\widehat{a} &= \left(\sum_{t=1}^{N} Y_{t-1}^2\right)^{-1}\left(\sum_{t=1}^{N} Y_t Y_{t-1}\right),\\
\widehat{b} &= \left(\sum_{j=1}^{N} Y_{j-1}^2\right)\left[\sum_{t=1}^{N}\left(\widehat{R}_t^2+W_{t-1}^2(\widehat{a})-2mT_{t-1}\right)Z_{t-1}T_{t-1}\right]\cdot I^{-1}-\\
&\quad-\left(\sum_{t=1}^{N} Z_{t-1}T_{t-1}Y_{t-1}\right)\left[\sum_{j=1}^{N}(\widehat{R}_j^2+W_{j-1}^2(\widehat{a})-2mT_{j-1})Y_{j-1}\right]\cdot I^{-1},\\
I &= \left(\sum_{t=1}^{N} Z_{t-1}^2T_{t-1}^2\right)\left(\sum_{j=1}^{N} Y_{j-1}^2\right)-\left(\sum_{j=1}^{N} Z_{t-1}T_{t-1}Y_{t-1}\right)^2,\\
\widehat{c} &= \left[\widehat{b}\sum_{t=1}^{N} Z_{t-1}T_{t-1}Y_{t-1}-\sum_{t=1}^{N}\left(\widehat{R}_t^2+W_{t-1}^2(\widehat{a})-2mT_{t-1}\right)Y_{t-1}\right]\cdot\\
&\quad\cdot\left(2m\sum_{t=1}^{N} Y_{t-1}^2\right)^{-1}
\end{aligned}
$$

where

$N+1$ is the size of the sample	$\theta_t = m(U_t+V_t-1)+V_t(E_t-m)$
$Y_t = Y_t - m$	$W_t(\widehat{a}) = \widehat{a}Y_t + m$
$\widehat{R}_t = \widehat{B}_t Y_{t-1} + \theta_t$	$T_t = Y_t + m$
$\widehat{B}_t = U_t - \widehat{a}$	$Z_t = Y_t - m$

It was proved in [7] that vectors $\mathbb{D}=(a,b,c)$ and $\widehat{\mathbb{D}}=(\widehat{a},\widehat{b},\widehat{c})$ are such that the vector $\widehat{\mathbb{D}}$ converges almost surely to the vector $\mathbb{D}$, and the vector $\sqrt{N}(\widehat{\mathbb{D}}-\mathbb{D})$ has the distribution which converges to the normal distribution with zero mean. Hence, the estimates $\widehat{a}$, $\widehat{b}$ and $\widehat{c}$ are consistent (and strong consistent) and asymptotically unbiased.

Now using the method of moments we shall show what are the possibilities of estimating parameters of some special types of the first order autoregressive time series with exponential $\mathcal{E}(\lambda)$ marginal distribution.

2. Estimating of Parameters of AREX(1)

AREX(1) [4] is one of the most general models of the first order autoregressive time series with exponential marginals. Its primary definition was

(2.1)
$$X_t = \begin{cases} \delta_t & w.p. \quad p_0 \\ \alpha X_{t-1} & w.p. \quad p_1 \\ \beta X_{t-1} + \delta_t & w.p. \quad q_1 \end{cases}, \quad \delta_t = \begin{cases} E_t & w.p. \quad B_0 \\ \alpha E_t & w.p. \quad B_1 \\ p_0\beta(p_0+q_1)^{-1}E_t & w.p. \quad B_2(p_0+q_1)^{-1} \end{cases}$$

where

(2.2)
$$\begin{gathered} p_0 + p_1 + q_1 = 1, \quad 0 \le p_1 + p_0\beta \le \alpha \le \beta < 1, \\ B_0 = \frac{1-\beta}{p_0+q_1-p_0\beta}, \quad B_1 = \frac{p_1(\beta-\alpha)}{\alpha(p_0+q_1)-p_0\beta}, \\ B_2 = \frac{\beta q_1(\alpha - p_1 - p_0\beta)(p_0+q_1)}{(p_0+q_1-p_0\beta)[\alpha(p_0+q_1)-p_0\beta]}. \end{gathered}$$

Using the difference equation (1.1) and the definition of its random coefficients U_t and V_t given by Popović [6], AREX(1) will be defined as

$$X_t = U_t X_{t-1} + V_t E_t$$

where

(2.3)
$$\begin{aligned} & P(U_t = 0) = p_0, \quad P(U_t = \alpha) = p_1, \quad P(U_t = \beta) = q_1 \\ & P(V_t = 0) = p_1, \ P(V_t = \alpha) = (p_0+q_1)B_1, \ P\left(V_t = \frac{p_0\beta}{p_0+q_1}\right) = B_2, \\ & P(V_t = 1) = (p_0+q_1)B_0 \\ & P((U_t,V_t) = (0,\alpha)) = p_0\beta_1, \ P\left((U_t,V_t) = \left(0, \frac{p_0\beta}{p_0+q_1}\right)\right) = \frac{p_0B_2}{p_0+q_1}, \\ & P((U_t,V_t)) = (0,1)) = p_0B_0, \ P((U_t,V_t) = (\alpha,0)) = p_1, \\ & P((U_t,V_t) = (\beta,\alpha)) = q_1B_1, \ P\left((U_t,V_t) = \left(\beta, \frac{p_0\beta}{p_0+q_1}\right)\right) = \frac{q_1B_2}{p_0+q_1}, \\ & P((U_t,V_t) = (\beta,1)) = q_1B_0 \end{aligned}$$

under the conditions (2.2).

So, the first and the second moment of the random variables U_t are

(2.4)
$$\begin{aligned} a &= E(U_t) = \alpha p_1 + \beta q_1 \\ b &= E(U_t^2) = \alpha^2 p_1 + \beta^2 q_1 \end{aligned}$$

It is clear that $\alpha < \dfrac{b}{a} < \beta$ and $\alpha\beta + b \ge a(\alpha+\beta)$. So the system of equations (2.4) permits us to solve p_1 and q_1 in the unique way and consequently p_0. So, if λ, α and β are known, p_0, p_1 and q_1 can be unequally estimated in the following way

(2.5)
$$\widehat{p}_0 = \frac{\alpha\beta + \widehat{b} - \widehat{a}(\alpha+\beta)}{\alpha\beta}, \ \widehat{p}_1 = \frac{\widehat{a}\beta - \widehat{b}}{\alpha(\beta-\alpha)}, \ \widehat{q}_1 = \frac{\widehat{b} - \widehat{a}\alpha}{\beta(\beta-\alpha)}.$$

Meanwhile, if α or β, or both of them are not known, standard procedure suposses using the other moments, preciously

$$c = E(V_t^2) \quad \text{or} \quad 1 - b - c = E(U_t V_t). \tag{2.6}$$

But, if we use, for instance, the last equation, we will have

$$\frac{\beta^5 P_3(\alpha) + \beta^4 P_4(\alpha) + \beta^3 P_5(\alpha) + \beta^2 P_6(\alpha) + \beta P_7(\alpha) + Q(\alpha)}{\beta^5 R_4(\alpha) + \beta^4 R_3(\alpha) + \beta^3 R_6(\alpha) + \beta^2 R_7(\alpha) + \beta S_7(\alpha) + S_5(\alpha)} = 1 - b - c, \tag{2.7}$$

where P_i, R_i, Q_i and S_i are polinoms of the degree i.

The same situation is with the other equation.

So, the general problem of estimating parameters of the model AREX(1) is unsolvable in this way. In spite of that in special cases the use is evident.

3. Special Cases

3.1. If we set $p_0 = 0$ and $0 < p_1 < 1$ in (2.1), we shall have so called FAREX(1) [4]. That means that distributions of random variables U_t, V_t and (U_t, V_t) in (1.1) will be

$$P(U_t = \alpha) = 1 - P(U_t = \beta) = p_1$$

$$P(V_t = 0) = \beta - \frac{p_1(\beta - \alpha)}{\alpha},\ P(V_t = \alpha) = \frac{p_1(\beta - \alpha)}{\alpha},\ P(V_t = 1) = 1 - \beta$$

$$P((U_t, V_t) = (\alpha, 0)) = p_1,\ P((U_t, V_t) = (\beta, 0)) = \frac{(\alpha - p_1)\beta}{\alpha},$$

$$P((U_t, V_t) = (\beta, 1)) = 1 - \beta,\ P((U_t, V_t) = (\beta, \alpha)) = \frac{p_1(\beta - \alpha)}{\alpha}$$

It has been proved in [7] that after we have got the estimates $\widehat{a}$, $\widehat{b}$ and $\widehat{c}$ by means of the method of least squares in two steps, we shall have unique estimates of p_1, α and β applying the method of moments.

The unique estimates are:

$$\widehat{p}_1 = \frac{(\widehat{a} + \widehat{b} + \widehat{c} - 1 - \widehat{a}^2)^2}{(\widehat{a} - \widehat{a}^2 + \widehat{b} + \widehat{c} - 1)^2 + (1 - \widehat{a})^2(\widehat{b} - \widehat{a}^2)},\ \widehat{\alpha} = \frac{\widehat{b} - \widehat{a} + \widehat{a}\widehat{c}}{\widehat{a} + \widehat{b} + \widehat{c} - 1 - \widehat{a}^2},$$

$$\widehat{\beta} = \frac{1 - \widehat{b} - \widehat{c}}{1 - \widehat{a}}.$$

3.2. If we set $\alpha = p_1 = 0$ in (2.1), we shall have well known NEAR(1) [3]. In the light of difference equation (1.1), its definition includes the following distributions

$$P(U_t = \beta) = 1 - P(U_t = 0) = q_1,\ 0 < q_1 \le 1,\ 0 < \beta \le 1,\ q_1\beta = 1$$

$$P(V_t = 1) = 1 - P(V_t = \beta(1 - q_1)) = \frac{1 - \beta}{1 - (1 - q_1)\beta}$$

$$P((U_t, V_t) = (0, \beta(1 - q_1)) = (1 - q_1)\left[1 - \frac{1 - \beta}{1 - (1 - q_1)\beta}\right],$$

$$P((U_t, V_t) = (0, 1)) = (1 - q_1)\frac{1 - \beta}{1 - (1 - q_1)\beta},$$

$$P((U_t, V_t) = (\beta, (1 - q_1)\beta) = q_1\left[1 - \frac{1 - \beta}{1 - (1 - q_1)\beta}\right],$$

$$P((U_t, V_t) = (\beta, 1)) = q_1\frac{1 - \beta}{1 - (1 - q_1)\beta}.$$

There are only two possible unknown parameters β and q_1. So, it is enough to have only two estimated moments

$$a = \beta q_1 \quad \text{and} \quad b = \beta^2 q_1\,, \quad \text{ie } \widehat{a} \text{ and } \widehat{b}$$

to solve the estimates $\widehat{\beta}$ and $\widehat{q_1}$:

$$\widehat{\beta} = \frac{\widehat{b}}{\widehat{a}}\,, \qquad \widehat{q_1} = \frac{\widehat{a}^2}{\widehat{b}}.$$

It is easy to verify that ratios $\widehat{b}/\widehat{a}$ and $\widehat{a}^2/\widehat{b}$ setisfy the conditions of the definition of NEAR(1) process.

3.3. If in (2.1) we set $\alpha = \beta$, we shall define so called SAREX(1) [4]. Then distributions of U_t, V_t and (U_t, V_t) will be:

$$\begin{aligned}
&P(U_t = 0) = 1 - P(U_t = \alpha) = p_0\\
&P(V_t = 0) = p_1,\ P\left(V_t = \frac{\alpha p_0}{p_0 + q_1}\right) = B_2,\ P(V_t = 1) = (p_0 + q_1)B_0\\
&1 - P\left((U_t, V_t) = \left(\alpha, \frac{\alpha p_0}{p_0 + q_1}\right)\right) = P((U_t, V_t) = (\alpha, 1)) = q_1 B_0
\end{aligned}$$

The unique estimates of α and p_0 according to the relations

$$a = \alpha(1 - p_0), \qquad b = \alpha^2(1 - p_0),$$

using the least square estimatiors for a and b, will be

$$\widehat{\alpha} = \frac{\widehat{b}}{\widehat{a}}\,, \qquad \widehat{p_0} = \frac{\widehat{b} - \widehat{a}^2}{\widehat{b}}.$$

But the last parametar of this model, q_1, is conected with the second moment c

$$c = E(V_t^2) = \frac{\alpha^2 p_0^2 [q_1 - (1-\alpha)(1-p_0)]}{(p_0 + q_1)[p_0(1-\alpha) + q_1]} + \frac{(1-\alpha)(p_0 + q_1)}{p_0(1-\alpha) + q_1}\,.$$

If $q_1 \neq p_0$ and $q_1 \neq p_0(1-\alpha)$, the only information about q_1 is the quadratic equation

$$\frac{\dfrac{a-b}{a} q_1^2 + \left[\dfrac{(b-a^2)^2}{a^2} + 2\dfrac{(a-b)(b-a^2)}{ab}\right] q_1 + \dfrac{(a-b)(b-a^2)^2(1-b^2)}{ab^3}}{q_1^2 + \left[\dfrac{(b-a^2)(2a-b)}{ab}\right] q_1 + \dfrac{(b-a^2)^2(a-b)}{ab^2}} = c$$

and its solution in respect of q_1 may not be real without disturbing the definition of SAREX(1).

3.4. Taking (2.1) as a starting point once more, we can choose TEAR(1) [3] in two possible ways:

I	$\beta = 1,$	$\alpha = p_0 + p_1,$	$p_0 + q_1 = 1$
II	$\beta = 1,$	$p_1 = 0,$	$p_0 + q_1 = 1.$

It follows that

$$P(U_t = 1) = 1 - P(U_t = 0) = q_1$$
$$P(V_t = 1 - q_1) = 1$$
$$P((U_t, V_t) = (1, 1 - q_1)) = 1 - P((U_t, V_t) = (0, 1 - q_1)) = q_1$$

Obviously that the only unknown parameter will be q_1 and it will be estimated as

$$\widehat{q_1} = \widehat{a}$$

4. Discussion and Conclusion

The mixture of methods for estimating parameters which was applied above is one of the most general methods for estimating parameters of the first order autoregressive time series with exponential $\mathcal{E}(\lambda)$ marginal distribution. Some other analitical methods presented, for instance, by Gaver and Lewis [2], Chiaw H.Sim [8] or Billard and Mohamed [1] can not be applied. The last one, for instance, can not be applied whenever X_t does not depend on δ_t with som possitive probability p_1. The method of Gaver and Lewis can not be applied whenever X_t is not pure autoregression, ie when $p_0 > 0$ and so on.

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