Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 6:2 (1992), 329–332

FILOMAT '92, Niš, October 8–10, 1992

ABOUT lAZZERINI'S EXPERIMENT

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ABSTRACT. *We are considering Lazzerini's success in Buffon's needle experiment. Especially, we find the probability of the same or better result and the expecting number of trials. Some unexpected results are obtained and explained.*

Problems. In Buffon's needle experiment [1], Lazzerini [4] was throwing a needle, of a relative length 5/6, on a set of parallel lines $3408 = 213 \cdot 16$ times and got $1808 = 113 \cdot 16$ sections. Such result gave him the 'Chinese' approximation $\pi \approx 355/113$ (the error is less then $3 \cdot 10^{-7}$). Of course, he obtained such precision because he was throwing a needle until he reached it.

We shall find a probability that one can repeat his result with the same or less number of trials, and an expecting value for the number of trials. At the end, we shall consider a small modification of the experiment.

Basic formulas. Let us denote $l = 5/6$, $a = 213$, $b = 113$ and $c = a - b = 100$. The probability of a section in a single trial is $p = 2l/\pi = 5/(3\pi)$ and the probability that in a sequence of $a \cdot n$ trials we have $b \cdot n$ sections is

$$p_n = \binom{an}{bn} p^{bn} q^{cn} \quad (q = 1 - p). \tag{1}$$

Let τ be the first n for which in the sequence of the first $a \cdot n$ trials we have $b \cdot n$ sections. As usual, $\tau = +\infty$ if there are not such sequence. Firstly, we should find the distribution of τ.

Denote $\tilde{p}_n = P(\tau = n)$ and $\tilde{q} = P(\tau = +\infty)$. Using the generating functions and the method of recurrent events [2, Chapter XIII], we shall find recurrent equalities

$$\tilde{p}_1 = p_1, \quad \tilde{p}_n = p_n - \sum_{k=1}^{n-1} p_k \tilde{p}_{n-k} \tag{2}$$

and an explicit expression

$$\tilde{q} = \frac{1}{1 + \sum_1^\infty p_n}.$$

1991 Mathematics subject classification: 60D05

Supported by Grant 0401A of FNS through Math. Inst. SANU

Since $P(\tau = +\infty) = \tilde{q}$ could be different from zero, $E(\tau)$ need not exist. Because of that, we should find the conditional expectation $E(\tau \mid \tau < +\infty)$. Again, using the generating functions and the method of recurrent events,

$$E(\tau \mid \tau < +\infty) = \sum_1^\infty n \frac{\tilde{p}_n}{1-\tilde{q}} = \frac{\sum_1^\infty n p_n}{(1+\sum_1^\infty p_n)(\sum_1^\infty p_n)}.$$

Numeric evaluations. From (1) and (2), we can calculate the probabilities $\tilde{p}_n$ and, especially, the probability that we can repeat Lazzerini's result with $a \cdot n$ ($n \le 16$) trials, i. e. $P(\tau \le 16) = \sum_1^{16} \tilde{p}_n$. Since an and bn are large, it is useful to approximate the probabilities p_n. First, since the probability of a section in a single trial $p = 5/(3\pi)$ is near to $b/a = 113/213 = (5 \cdot 113)/(3 \cdot 355)$, let

$$\epsilon = \frac{ap}{b} - 1 \approx 8.4913678 \cdot 10^{-8}.$$

(In each numeric result, all digits will be significant.)

Using the well known Stirling's formula, we have

$$p_n \sim k_1 n^{-1/2} e^{-k_2 n}\left(1 - \frac{k_3}{n}\right) \tag{3}$$

(the term k_3/n is for better precision of calculations), where

$$\begin{aligned} k_1 &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{a}{bc}} \approx 0.05477226, \\ k_2 &= -b \ln\frac{ap}{b} - c\ln\frac{aq}{c} \\ &= \frac{b(b+c)}{2c}\epsilon^2 + \frac{b(b^2-c^2)}{3c^2}\epsilon^3 + \frac{b(b^3+c^3)}{4c^3}\epsilon^4 + \cdots \\ &\approx \frac{ab}{2c}\epsilon^2 \approx 8.677275 \cdot 10^{-13}, \\ k_3 &= \frac{a^2 - bc}{12abc} \approx 0.001179560. \end{aligned}$$

(Note the extremely small k_2.)

Now, from (2) and (3) we can evaluate

$$\tilde{p}_1 \approx 0.05471, \ldots \; \tilde{p}_{16} \approx 0.00822,$$

and

$$P(\tau \le 16) = \sum_1^{16} \tilde{p}_n \approx 0.28873.$$

For $\tilde{q}$ and $E(\tau \mid \tau < +\infty)$, it is necessary to summarize the series. First,

$$\begin{aligned} \sum_1^\infty p_n &\approx \int_0^\infty p(x)\,dx \approx k_1 \int_0^\infty x^{-1/2} e^{-k_2 x}\,dx = k_1 \frac{\Gamma(1/2)}{k_2^{1/2}} \\ &\approx \frac{1}{b\epsilon} \approx 1.04218 \cdot 10^5 \end{aligned}$$

where $p(x) = p_n$ (for $n - 1 < x \le n$) and $\Gamma(\cdot)$ is the gamma function. The errors of the approximations (for the first three '$\approx$') are, respectively,

$$\mathrm{err}_1 \le \int_0^1 p_x \, dx \le k_1 \int_0^1 x^{-1/2} \, dx = 2k_1 < 0.2,$$

$$\mathrm{err}_2 \approx \sum_1^\infty k_1 n^{-1/2} e^{-k_2 n} \frac{k_3}{n} < k_1 k_3 \sum_1^\infty n^{-3/2} < 3k_1 k_3 < 2 \cdot 10^{-4},$$

$$\mathrm{err}_3 \approx \frac{1}{b\epsilon} \cdot \frac{1}{2} \cdot \frac{2(b-c)}{3c} \epsilon = \frac{b-c}{3bc} < 4 \cdot 10^{-4},$$

so that all digits are significant.

Similarly,

$$\sum_1^\infty n p_n \approx \int_0^\infty x p(x) \, dx \approx k_1 \int_0^\infty x^{1/2} e^{-k_2 x} \, dx = k_1 \frac{\Gamma(3/2)}{k_2^{3/2}}$$

$$\approx \frac{bc}{a} \frac{1}{(b\epsilon)^3} \approx 6.00524 \cdot 10^{16}$$

with the possible errors of the approximations

$$\mathrm{err}_1 \le \int_0^1 x p_x \, dx \le k_1 \int_0^1 x^{1/2} \, dx = \frac{2k_1}{3} < 0.04,$$

$$\mathrm{err}_2 \approx \sum_1^\infty n k_1 n^{-1/2} e^{-k_2 n} \frac{k_3}{n} < k_1 k_3 \left[1 + \int_1^\infty x^{-1/2} e^{-k_2 x} \, dx \right]$$

$$< k_1 k_3 \left(1 + \frac{\Gamma(1/2)}{k_2^{1/2}} \right) = k_3 \left(k_1 + \frac{1}{b\epsilon} \right) < 200,$$

$$\mathrm{err}_3 \approx \frac{bc}{a} \frac{1}{(b\epsilon)^3} \cdot \frac{3}{2} \cdot \frac{2(b-c)}{3c} \epsilon = \frac{b-c}{ab^2\epsilon^2} < 7 \cdot 10^8.$$

Again, all digits are significant.

Finally,

$$P(\tau = +\infty) = \tilde{q} \approx \frac{1}{1 + \frac{1}{b\epsilon}} = \frac{b\epsilon}{1 + b\epsilon} \approx 0.95952 \cdot 10^{-5},$$

$$E(\tau \mid \tau < +\infty) \approx \frac{bc}{a} \frac{1}{b\epsilon(1 + b\epsilon)} = \frac{c}{a\epsilon(1 + b\epsilon)} \approx 5.5289 \cdot 10^6,$$

with all significant digits.

Conclusions. The probability that we never obey Lazzerini's result, i. e. $P(\tau = +\infty)$, is extremely small, but not zero. The expecting time (in steps of $a = 213$ trials), giving that τ is finite, i. e. $E(\tau \mid \tau < +\infty)$ is extremely large. So, it seems that Lazzerini's result is improbable. But, the probability that we have his result with the same or less number of trials, i. e. $P(\tau \le 16)$, is almost 29%, what seems like a paradox.

An explanation. To explain the 'paradox' we shall change the condition of the experiment. Let us call the old experiment $\mathcal{E}$ and the new one $\mathcal{E}_0$. In $\mathcal{E}_0$ we shall leave $a = 213$ and $b = 113$, but

$$l = \frac{113\pi}{355} \cdot \frac{5}{6} = \frac{b\pi}{2a}$$

so that $p = 2l/\pi = b/a$. Then, since $p = (bn)/(an)$, we can get the correct value for π in $a \cdot n$ trials. (Note that an experiment with $l = \pi/4$, $a = 2$, $b = 1$ is more effective [3].)

In the same way as before, $\epsilon = 0$, $k_2 = 0$ and

$$p_n \sim k_1 n^{-1/2} \left(1 - \frac{k_3}{n}\right).$$

where k_1 and k_3 are the same as in $\mathcal{E}$.

Since, in $\mathcal{E}$, k_2 is very small, and, in $\mathcal{E}_0$, it is zero, the distribution for τ is not the same, but, for smaller n, the differences will be very small. Because of that, it is not a surprise that numeric evaluations of the probabilities $\tilde{p}_n$ give the same result as in $\mathcal{E}$:

$$\tilde{p}_1 \approx 0.05471, \dots \tilde{p}_{16} \approx 0.00822,$$

$$P(\tau \leq 16) = \sum_1^{16} \tilde{p}_n \approx 0.28873.$$

The series $\sum_1^\infty p_n$ and $\sum_1^\infty np_n$ are divergent, but, using the same methods as before, we can find

$$P(\tau = +\infty) = 0,$$

$$E(\tau \mid \tau < +\infty) = E(\tau) = +\infty.$$

So, now we have a proper random variable with an infinity expecting value. Of course, here does not exist any paradox. Note that the results of the experiment $\mathcal{E}_0$ are very similar to the results of the coin tossing experiment in [2, Chapter III].

Now, we can consider $\mathcal{E}$ as an approximation of the 'limit' experiment $\mathcal{E}_0$ (in a sense that the distribution of τ in $\mathcal{E}$ tends, when $\epsilon \to 0$, to the distribution of τ in $\mathcal{E}_0$). Then, the results in $\mathcal{E}$ are approximations of the appropriate results in $\mathcal{E}_0$ and the paradox does not exist.

References

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