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A LIMIT PROBLEM FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH THE COEFFICIENTS HAVING RANDOM INTEGRAL CONTRACTORS

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Abstract. *In this paper a limit problem for stochastic differential equations of Ito type depending on a standard Wiener process and on a random Poisson measure is investigated. It is shown that the Lipschitz condition is equivalent with the existence of bounded random integral contractors for coefficients of these iquations.*

Introduction

Before stating the main results to be proved in the present paper, we first list the necessary preliminaries and assumptions needed in our discussion.

Throught the paper, we suppose that all random variables and processes considered here are defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. In general, for a stochastic process $x(t)$, denote by $\mathcal{F}_t(x)$ the least σ-algebra for which $x(s)$ is measurable for every $s \in [0,t]$.

Let $\mathcal{B}$ be the σ-algebra on R and $\prod(A)$, $A \in \mathcal{B}$, be a positive bounded measure on the space $(R, \mathcal{B})$. Denote by $p(t,A) = t \cdot \prod(A)$, $t \geq 0$, a random Poisson measure with a parameter $t \cdot \prod(A)$, and $\tilde{p}\,(t,A) = p(t,A) - t \cdot \prod(A)$ the corresponding centered Poisson measure. Also, let $\{W(t),\, t \geq 0\}$ be a standard Wiener process not depending on the Poisson measure and let η be a random variable not depending on the both ones. In the usual way, the Wiener process and the Poisson measure generate the family of σ-algebras $(\mathcal{F}_t,\, t \geq 0)$, in the sence that $\mathcal{F}_t$ is the least σ-algebra for which $W(t)$ and $\tilde{p}\,(s,A)$ are measurable for every $s \in [0,t]$.

Following the traditions of the classical theory of stochastic differential equations (shorter SDE), we require that non-random functions

$$f_i : [0,T] \times R \to R, \quad i = 1,2, \qquad f_3 : [0,T] \times R \times R \to R,$$

T =const>0, are measurable on corresponding σ-algebras on their domains, satisfy the uniform Lipschitz condition on the second argument and the condition of the restriction on growth; i.e. there exists a constant $L > 0$, such that for all $t \in$

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$[0,T]$, $x, y \in R$, we have

$$(1.1)\quad \begin{aligned} &|f_1(t,x)-f_1(t,y)|^2+|f_2(t,x)-f_2(t,y)|^2+\\ &+\int_R |f_3(t,x,u)-f_3(t,y,u)|^2 \prod(du) \le L|x-y|^2,\\ &|f_1(t,x)|^2+|f_2(t,x)|^2+\int_R |f_3(t,x,u)|^2\prod(du)\le L(1+|x|^2).\end{aligned}$$

Under these conditions and if for any $x \in R$

$$(1.2)\quad \int_0^T |f_1(t,x)|\,dt<\infty,\ \int_0^T |f_2(t,x)|^2\,dt<\infty,\ \int_0^T\int_R |f_3(t,x,u)|^2\prod(du)<\infty,$$

it is well-known that the SDE

$$(1.3)\quad \begin{aligned} &d(x)=f_1(t,x(t))dt+f_2(t,x(t))dW(t)+\int_R f_3(t,x(t),u)\tilde{p}(dt,du), \qquad t\in(0,T],\\ &x(0)=\eta \qquad \text{almost surely}\end{aligned}$$

has an unique solution $x(t)$, $t \in [0,T]$, as a stochastoc process adapted to the family of σ-algebras $(\mathcal{F}_t,\ t\ge 0)$, which trajectories are right continuous and have left hand limits almost surely (see, for instance, [2]).

In the paper [3] the sequence of SDEs

$$(1.4)\quad \begin{aligned} dx(t)&=f_1^n(t,x_n(t))dt+f_2^n(t,x_n(t))dW(t)+\\ &\quad+\int_R f_3^n(t,x_n(t),u)\tilde{p}(dt,du),\quad t\in(0,T],\\ x_n(t)&=\eta \qquad \text{almost surely},\ n\in N,\end{aligned}$$

is considered and the following theorem is proved.

THEOREM 1.1. *If the functions f_i^n, $i=1,2,3$, $n\in N$, satisfy the same conditions as the functions f_i, $i=1,2,3$ (the conditions (1.1) and (1.2)), and the condition*

$$(1.5)\quad \begin{aligned} &\sum_{n=1}^{\infty} \sup_{t,x}\{|f_1(t,x)-f_1^n(t,x)|^2+|f_2(t,x)-f_2^n(t,x)|^2-\\ &-\int_R |f_3(t,x,u)-f_3{}^n(t,x,u)|^2\prod(du)\}<\infty,\end{aligned}$$

then the sequence of solutions $\{x_n\}$, $n\in N$, of the SDEs (1.4) *converges almost surfely, uniformly on $[0,T]$, to the solution x of the* SDE (1.3).

The primary question in this problem, and in the other analogous problems, is that: could the uniform Lipschitz condition for the functions f_i, f_i^n, $i = 1, 2, 3$, $n \in N$, be changed with some more general ones; for instance, with a condition of existence of bounded random integral contractors for these functions. Note that the concept of integral contractors was introduced by Altman ([1]) as a useful tool for studying different classes of deterministic equations in Banach spaces. Later, this approach was appropriately used to analysis of special classes of stochastic differential equations (see, for instance, [5], [6], [9]). Immediately, it is utilised to obtain very general conditions for the existence and uniqueness of their solutions.

Becausse of the present problem, we have to give some necessary definitions and conclusions involving the concept of bounded random integral contractor for the SDE (1.3), itroduced in the paper [8], or, in some complicated sense, in [4].

Let D be the collection of real-valued stochastic processes defined on $[0, T]$, adapted with respect to the family of σ-algebras $(\mathcal{F}_t, t \geq 0)$, which trajectories are right continuous and have left hand limits almost surely.

Let $G_i : [0, T] \times R \to R$, $i = 1, 2$, $G_3 : [0, T] \times R \times R \to R$, be bounded measurable functions. For arbitrary $x, y \in D$, denote by

$$z(t) = y(t) + \int_0^t G_1(s, x(s))y(s)ds + \int_0^t G_2(s, x(s))y(s)dW(s) + \tag{1.6}$$
$$+ \int_0^t \int_R G_3(s, x(s), u)y(s)\tilde{p}(ds, du).$$

Also, we suppose that the functions f_i, $i = 1, 2, 3$, satisfy the following condition: let there exists a positive constant K, such that for all $t \in [0, T]$, $x \in D$, the inequality

$$\begin{aligned} &|f_1(t, x(t) + z(t)) - f_1(t, x(t)) - G_1(t, x(t))y(t)|^2 + \\ &|f_2(t, x(t) + z(t)) - f_2(t, x(t)) - G_2(t, x(t))y(t)|^2 + \\ &\int_R |f_3(t, x(t) + z(t), u) - f_3(t, x(t), u) - G_3(t, x(t), u)y(t)|^2 \prod(du) \\ &\leq K \sup_{s \in [0,t]} |y(s)|^2 \end{aligned} \tag{1.7}$$

holds almost surely.

DEFINITION 1.1. If the condition (1.7) is satisfied, then the triplets of functions (f_1, f_2, f_3) has a bounded random integral contractor, denoted by

$$\left\{ I + \int_0^t G_1 ds + \int_0^t G_2 dW(s) + \int_0^t \int_R G_3 \tilde{p}(ds, du) \right\}.$$

DEFINITION 1.2. A bounded random integral contractor is said to be regular, if the linear SDE (1.6) has at least one solution y in D for any x and z in D

DEFINITION 1.3. A function $h : [0,T] \times R \to R$ is said to be stochastically closed if for any x_n, $n \in N$, and x, y in D, such that $x_n \to x$ and $h(\cdot, x_n) \to y$ in $L^2([0,T] \times \Omega)$ sence, we have $y(t) = h(t, x(t))$ for every $t \in [0,T]$ almost surely.

It can be proved that if the functions f_i, $i = 1,2,3$, satisfy the Lipschitz condition on the second argument, then they are stochastically closed and have a regular bounded random integral contractor with $G_i = 0$, $i = 1,2,3$.

Using the preceding definition and conditions, in the paper [8] and in [4], in some complicated form, the theorem of existence and uniqueness of solution of the SDE (1.3) is stated and proved.

THEOREM 1.2. *If the functions f_i, $i = 1,2,3$, from the* SDE (1.3) *satisfy the condition* (1.2), *are stochastically closed and have a bounded random integral contractor, then the* SDE (1.3) *has a solution x in D. Moreover, if the bouded random integral contractor is regular, then the solution x in D is unique almost surely.*

2. Main results

In order to solve our main problem, we shall prove that the existence of a bounded random integral contractor is equivalent with a Lipschitz condition. The basic idea comes down from the paper [7], where the same problems for ordinary differential equations and for SDEs of Ito type (without details) are investigated. First, we shall formulate the following lemma.

Lemma 2.1. *Let $G_i : [0,T] \times R \to R$, $i = 1,2$, $G_3 : [0,T] \times R \times R \to R$, be bounded measurable functions. Then for every $x, y \in D$ there exists an unique solution $y \in D$ of the* SDE (1.6). *Furthemore, there exists a constant $\bar{L} > 0$, independent on x and z, such that*

$$E \sup_{t \in [0,T]} |y(t)|^2 \leq \bar{L} \cdot E \sup_{t \in [0,T]} |z(t)|^2.$$

PROOF. Taking into consideration the assumptions about the functions G_i, $i = 1,2,3$ (remind that they are measurable and bounded), it is not difficult to conclude that the coefficients of the linear SDE (1.6) satisfy with probability one the uniform Lipschitz condition and the condition of the restraction on growth. Therefore, for arbitrary $x, y \in D$, it follows the existence and uniqueness of solution $y \in D$ of this equation (see, for instance, [2]).

Suppose that

$$(2.1) \qquad |G_1(t,x)|^2 + |G_2(t,x)|^2 + \int_R |G_3(t,x,u)|^2 \prod(du) \leq M \text{ on } [0,T] \times R.$$

Rewritte the SDE (1.6) as

$$y(t) = z(t) - \int_0^t G_1(s,x(s))y(s)ds - \int_0^t G_2(s,x(s))y(s)dW(s) -$$
$$\int_0^t \int_R G_3(s,x(s),u)y(s)\tilde{p}(ds,du).$$

Using the Cauchy-Schwartz inequality and some basic properties of the Lesbegue and the Ito integrals, we obtain

$$E|y(t)|^2 \leq 4E \sup_{t\in[0,T]} |z(t)|^2 + 4M(T+2) \int_0^t E|y(s)|^2 ds,\ t \in [0,T].$$

By a version of the well-known Gronwall-Bellman's inequality, we have

$$E|y(t)|^2 \leq A \cdot E \sup_{t\in[0,T]} |z(t)|^2,\ t \in [0,T], \tag{2.2}$$

where A is a corresponding constant. Since

$$\sup_{t\in[0,T]} |y(t)| \leq \sup_{t\in[0,T]} |z(t)| + \int_0^T |G_1(s,x(s))| \cdot |y(s)| ds +$$

$$+ \sup_{t\in[0,T]} \left| \int_0^t G_2(s,x(s))y(s)dW(s) \right| + \sup_{t\in[0,T]} \left| \int_0^t G_3(s,x(s),u)y(s)\tilde{p}(ds,du) \right|,$$

and if we apply the well-known Doob's martingale inequality, we obtain

$$E \sup_{t\in[0,T]} |y(t)|^2 \leq 4 \left\{ E \sup_{t\in[0,T]} |z(t)|^2 + TM \int_0^T E|y(s)|^2 ds + 8M \int_0^T E|y(s)|^2 ds \right\}.$$

Using the estimation (2.2), it follows

$$E \sup_{t\in[0,T]} |y(t)|^2 \leq 4[1 + AMT(T+8)] E \sup_{t\in[0,T]} |z(t)|^2 = L \cdot E \sup_{t\in[0,T]} |z(t)|^2.$$

So, the Lemma is proved.

REMARK 2.1. The existence and uniqueness of solution of the SDE (1.6) could be proved directly using the method of successive approximations, or the Banach fixed point theorem.

REMARK 2.2. From Lemma 2.1. we conclude that every bounded random integral contractor is a regular one. Becouse of this fact, from Theorem 1.1. it follows that the solution of the SDE (1.3) is unique almost surely.

We are now in a position to estabilish the following theorem which could be very useful for solving different problems.

THEOREM 2.1. *The triplets of functions* (f_1, f_2, f_3) *has a bounded random integral contractor if and only if the functions* f_1, f_2, f_3 *satisfy the Lipchitz condition* (1.1) *uniformly with respect to* $t \in [0,T]$.

PROOF. If the functions f_i, $i = 1,2,3$, satisfy the Lipschitz condition (1.1), then there exists a bounded random integral contractor for $G_i = 0$, $i = 1,2,3$.

Let $\left\{I + \int_0^t G_1 ds + \int_0^t G_2 dW(s) + \int_0^t \int_R G_3 \tilde{(ds, du)}\right\}$ be a bounded random integral contractor for the triplets (f_1, f_2, f_3). For arbitrary real numbers $x, z \in R$, $z \neq 0$, the stochastic processes $x(t) = x$, $z(t) = z$ almost surely, are elements on D. From Lemma 2.1. it follows that the linear SDE

$$z = y(t) + \int_0^t G_1(s,x)y(s)ds + \int_0^t G_2(s,x)y(s)dW(s) + \int_0^t \int_R G_3(s,x,u)y(s)\tilde{p}(ds,du).$$

has an unique solution $y \in D$, for which

$$E \sup_{t\in[0,T]} |y(t)|^2 \leq L \cdot z^2.$$

Using the relation (1.7) and because of the fact that the functions G_i, $i = 1,2,3$, are bounded as in (2.1), we have for every $t \in [0,T]$

$$\begin{aligned}
&|f_1(t,x+z) - f_1(t,x)|^2 + |f_2(t,x+z) - f_2(t,x)|^2 + \\
&\int_R |f_3(t,x+z,u) - f_3(t,x,u)|^2 \prod(du) \leq \\
&2\left\{|f_1(t,x+z) - f_1(t,x) - G_1(t,x)y(t)|^2 + |G_1(t,x)y(t)|^2\right\} + \\
&+2\left\{|f_2(t,x+z) - f_2(t,x) - G_2(t,x)y(t)|^2 + |G_2(t,x)y(t)|^2\right\} + \\
&2\left\{\int_R \left[|f_3(t,x+z,u) - f_3(t,x,u) - G_3(t,x,u)y(t)| + |G_3(t,x,u)y(t)|^2\right] \prod(du)\right\} \\
&\leq 2K \sup_{s\in[0,t]} |y(s)|^2 + 2M|y(t)|^2 \leq 2(K+M) \sup_{t\in[0,t]} |y(t)|^2 \text{ almost surely.}
\end{aligned}$$

Since on the left side we have a constant almost surely, and from the Lemma 2.1., it follows

$$\begin{aligned}
&|f_1(t,x+z) - f_1(t,x)|^2 + |f_2(t,x+z) - f_2(t,x)|^2 + \\
&\int_R |f_3(t,x+z,u) - f_3(t,x,u)|^2 \prod(du) \leq \\
&2(K+M) \cdot E \sup_{t\in[0,t]} |y(t)|^2 \leq 2(K+M)Lz^2.
\end{aligned}$$

Thus, the proof is complete.

Therefore, the limit problem considered in Introduction could be formulated and solved using the Theorem 2.1.

REMARK 2.3. From this lemma we conclude that if the functions f_i, $i = 1,2,3$, satisfy the Lipschitz condition with respect to $t \in [0,T]$, then for every bounded measurable functions G_i, $i = 1, 2, 3$, the triplets (f_1, f_2, f_3) has a bounded random integral contractor defined as stated above.

THEOREM 2.2. *Assume that:*

(i) *The functions* f_i, f_i^n, $i = 1, 2, 3$, $n \in N$, *satisfy the corresponding conditions* (1.2) *and all of them are stochastically closed;*

(ii) *The tripletses of functions* (f_1, f_2, f_3), $({f_1}^n, {f_2}^n, {f_3}^n)$, $n \in N$, *have bounded random integral contractors respectively*

$$\left\{ I + \int_0^T G_1 ds + \int_0^t G_2 dW(s) + \int_0^t \int_R G_3 \tilde{p}(ds, du) \right\},$$

$$\left\{ I + \int_0^T G_1^n ds + \int_0^t G_2^n dW(s) + \int_0^t \int_R G_3^n \tilde{p}(ds, du) \right\}, \ n \in N;$$

(iii) *The condition* (1.5) *is satisfied.*

Then the sequence of solutions $\{x_n\}$, $n \in N$, *of the* SDEs (1.4) *converges almost surely, uniformly on* $[0, T]$, *to the solution* x *of the* SDE (1.3).

PROOF. Since the conditions (i) and (ii) are satisfied, using the Theorem 1.2. and the Lemma 2.1., the existence and uniqueness of solutions of the SDEs (1.3) and (1.4) are guaranteed. From the Theorem 2.1. it follows that the functions f_i, f_i^n, $i = 1, 2, 3$, $n \in N$, satisfy the Lipschitz condition uniformly on $[0, T]$ (and also the condition of the restriction on growth, what is not difficult to show). Therefore, using the Theorem 1.1., the proof is complete.

It could be very interesting to compare speeds of convergences of the sequence of solutions $\{x_n\}$, $n \in N$, to the solution x, if the conditions of the Theorem 1.1. or of the Theorem 2.2., are satisfied. In fact, it could be interesting to study how the speed of convergence depends on a choice of integral contractors. Aslo, we could extend the results of this paper on other SDEs; for example, on SDEs involving stochastic integrals with respect to any continuous martingale and martingale measure, or on different classes of stochasic integrodifferential equations.

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