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ON STAR-COMPACT SPACES WITH A G_δ-DIAGONAL

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ABSTRACT. *We introduce a new concept - τ-extendability of topological properties, and prove that certain compactness-type properties are τ-extendable for uncountable τ. Answering D. B. Shakhmatov's question we present for any cardinal λ an example of a pseudocompact space X with G_δ-diagonal and $|X| \geq \lambda$. Under CH this space X can be made 2-pseudocompact which is stronger than being pseudocompact. A ZFC example of a 2-pseudocompact space which has no dense relatively countably compact subspace is also given.*

0. Introduction

In some situations the properties of spaces can be characterized by the properties of their considerably "large" parts. Thus a space is sometimes representable as an increasing family of its "good" subspaces, while on some step it appeares that a subspace coincide with a whole space which is therefore "good". The proofs of A. V. Arhangelskii's theorems on the power of first countable compact spaces [Arh_3], and on metrizability of countably compact spaces with point-countable base [Arh_1], [Arh_2] are classical examples of such results. Our notion of τ-extendability is an attempt to provide a general scheme for some techniques working in such situations. In section 1 we introduce the notion of τ-extendability and prove that compactness, countably compactness, pseudocompactness, and some other properties are τ-extendable for any uncountable cardinal τ. Later, we use this fact for constructing examples.

Any countably compact space with a G_δ-diagonal is metrizable [Cha], [Gru]. This is also true for pseudocompact spaces with a regular G_δ-diagonal [Reed_2], but not for all pseudocompact spaces with a G_δ-diagonal, since there exist nonmetrizable pseudocompact Moore spaces. Every pseudocompact Moore space is separable [Reed_1], [Reed_2] and hence of cardinality $\leq$ @. This led D. B. Shakhmatov to the question whether the cardinality of a pseudocompact space with a G_δ-diagonal can be arbitrarily large.

In section 3 we answer D. B. Shakhmatov's question by presenting for any cardinal λ a pseudocompact space X with G_δ-diagonal and $|X| \geq \lambda$. In fact, this space X was constructed in [Ber]. This construction, however, left some freedom in choosing certain subfamilies. We restrict this freedom by setting some additional conditions which provide G_δ-diagonal. And some more restrictions help us to make this space 2-pseudocompact (assuming CH).

Let us recall that for a cover γ of X and a subset $A \subset X$ the n-th star is defined inductively: $St^{n+1}(A,\gamma) = St(St^n(A,\gamma),\gamma)$. For $A = \{x\}$ we write $St^n(x,\gamma)$ instead of $St^n(\{x\},\gamma)$.

DEFINITION 0.1. [Mat_2] A space Y is *n-pseudocompact* (where n is a natural number), if for every open cover γ of Y the cover $\gamma^{(n)} = \{St^n(x,\gamma) : x \in Y\}$ has a finite subcover.

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DEFINITION 0.2. [vDRRT], [Reed$_2$], [Sar], [Ikg$_2$] A space Y is *n-starcompact* (where n is a natural number) if for every open cover γ of Y the cover $\{St^n(U,\gamma) : \in \gamma\}$ has a finite subcover.

In [Reed$_2$], [vDRRT] n-pseudocompact spaces are called strongly n-starcompact. Strongly n-starcompact implies n-starcompact implies strongly n+1-starcompact. For a Tychonoff space, countably compact is equivalent to 1-pseudocompact, and pseudocompact is equivalent to 2-starcompact and to (strongly) n-starcompact for any $n \geq 3$. Hence we obtain the following sequence of implications: countably compact implies 1-starcompact implies 2-pseudocompact implies pseudocompact. None of these implications can be reversed [Mat$_1$], [vDRRT]. So it is natural to ask whether there exist an example like in section 4 but 1-starcompact.

QUESTION 1. Does there exist a 1-starcompact Tychonoff space X with a G_δ-diagonal such that $|X| > @$?

This problem should be compared with Problem 3.1. from [Reed$_2$]: is each 1-starcompact Moore space compact. The cardinality of a pseudocompact space with a point-countable base can be arbitrarily large [Shk$_1$]. This leads to the following

QUESTION 2. Let X be a pseudocompact space with a G_δ-diagonal and with a point-countable base. Can the cardinality of X be arbitrarily large?

We say that a subspace Y of a space X is countably compact in X provided every infinite subset $A \subset Y$ has a limit point in X. A space X is *countably pracompact* if it has a dense subspace which is countaably compact in X. Countably pracompact spaces were studied in [Ber], [Mil], [Arh$_2$], [Mat$_2$] - [Mat$_4$]. The terms "dense countably compactness" and "weak countably compactness" were also used. Here we follow the terminology of [Arh$_2$]. It was noted in [Ber], [Mil] that countably pracompact implies pseudocompact, and the examples of pseudocompact non countably pracompact spaces were given. Later, the examples of pseudocompact spaces X in which all the sets A, $|A| \leq \tau$ are closed, discrete, and C^*-embedded, were constructed in [Shk$_2$], [Rez], [Mat$_1$] for arbitrary τ; it is clear that this property completely destroys countably compactness and even countably pracompactness. So we can suppose that the place of the countable pracompactness in the compactness type properties scale is fairly close to countably compactness and far from pseudocompactness. It has been noted in [Mat$_2$], that countably pracompact implies 2-pseudocompact.

In section 5 we give an example of a 2-pseudocompact space which is not countably pracompact.

QUESTION 3. Does there exist a 1-starcompact Tychonoff space which is not countably pracompact?

S. Mrówka's space $\Psi = \mathcal{N} \cup \mathcal{R}$ [Mru] is a classical example of a countably pracompact space which is not countably compact. It is also locally compact, separable, and Moore. It has been shown in [vDRRT] that Ψ is not 1-starcompact.

Ψ is constructed as follows: $\mathcal{N}$ is open and discrete in Ψ, $|\mathcal{N}| = \omega$, $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of $\mathcal{N}$, $|\mathcal{R}| \geq \omega_1$ (a family of sets is *almost disjoint* if the intersection of any two its distinct elements is finite). A basic neighbourhood of $r \in \mathcal{R}$ is $\{r\} \cup (r \setminus K)$ where K is arbitrary finite subset of r.

Since Ψ is a Moore space it has a G_δ-diagonal. But $|\Psi| \leq 2^{|\mathcal{N}|} = @$, and if a countable $\mathcal{N}$ is replaced by uncountable open discrete set then the space will not have G_δ-diagonal.

However, many examples are based on replacing isolated points from $\mathcal{N}$ by clopen compact subsets [Ber], [Sc], [vDRRT]. In our case, a special construction of a base will provide G_δ-diagonal for X.

The G_δ-diagonal example from section 3 has been constructed by the second author, while its 2-pseudocompact CH modification from section 4 and the example from section 5 were found by the first author. Recently, the authors have learnt from [Shk$_3$] that S. Watson had also constructed a pseudocompact space with G_δ-diagonal having arbitrary large cardinality.

1. τ-extendable properties

DEFINITION 1.1. A family $\mathcal{A}$ of subspaces of a space X is called *ω-conservative* provided for every subfamily $\mathcal{B} \subset \mathcal{A}$ and every point $x \in \overline{\cup\mathcal{B}}$ there exists a countable $\mathcal{C} \subset \mathcal{B}$ for which $x \in \overline{\cup\mathcal{C}}$.

DEFINITION 1.2. A space X satisfies the property $\mathcal{R}_\tau(\mathcal{P})$ where τ is infinite ordinal and $\mathcal{P}$ is a topological property if there exists an operation $P(\cdot)$ which defines for every $A \subset X, |A| \leq \tau$ a closed subspace $P(A) \subset X$ such that $A \subset P(A), P(A)$ satisfies $\mathcal{P}$ and the following conditions hold:

(1) $P(A) \subset P(B)$ if $A \subset B \subset X$ and $|B| \leq \tau$,
(2) the family $\{P(A) : A \subset X, |A| \leq \tau\}$ is ω-conservative,
(3) if $\mathcal{A}$ is a chain of subset of X, $|A| \leq \tau$ for any $A \in \mathcal{A}$, and $|\mathcal{A}| \leq \tau$ then $P(\cup\mathcal{A}) \subseteq \overline{\cup\{P(A) : A \in \mathcal{A}\}}$,
(4) $|P(A)| \leq \tau$ if $A \subset X$, and $|A| \leq \tau$.

DEFINITION 1.3. A topological property $\mathcal{P}$ is τ-extendable in a class $\mathcal{K}$ of spaces provided every $X \subset \mathcal{K}$ possesing $\mathcal{R}_\tau(\mathcal{P})$ satisfies $\mathcal{P}$. If $\mathcal{K}$ is the class of all Tychonoff spaces, we shall omit the words "in the class $\mathcal{K}$".

The following theorem will be used in section 6.

THEOREM 1.1. *n-pseudocompactness is τ-extendable for every uncountable cardinal τ and every $n \in \omega$.*

PROOF. Supose γ is an open cover of X, and $\gamma^{(n)}$ does not have a finite subcover, while X satisfies the property $\mathcal{R}_\tau(npc)$ for some infinite cardinal τ, where npc denotes n-pseudocompactness. Let us compose an increasing chain $\{X_\alpha : \alpha < \omega_1\}$ of subspaces of X for which $|X_\alpha| \leq \tau$ for all $\alpha < \omega_1$. Put $X_0 = \{x_0\}$ where $x_0 \in X$ is arbitrary point. Let $\alpha < \omega_1$, and the subspaces X_γ have been constructed for all $\gamma < \alpha$. Denote $\mathcal{K}_\alpha$ the family of all finite subsets of the set $M_\alpha = P(\cup\{X_\gamma : \gamma < \alpha\})$. For every $K \in \mathcal{K}_\alpha$ pick a point $x_K \in X \setminus St^n(K,\gamma)$ and put $X_\alpha = M_\alpha \cup \{x_K : K \in \mathcal{K}_\alpha\}$. It is clear, that $|M_\alpha| \leq \tau$, and thus $|\mathcal{K}_\alpha| \leq \tau$ and $|X_\alpha| \leq \tau$.

Put $X^* = \cup\{X_\alpha : \alpha < \omega_1\}$ and $P^* = P(X^*)$. As P^* is npc, there exists a finite $K^* \subset P^*$ for which $St^n(K^*,\gamma) \supset P^*$. By (3) from Definition 1.2., we have $K^* \subset P^* \subset \overline{\cup\{P(X_\alpha) : \alpha < \omega_1\}}$. The property (2) implies that for every $k \in K^*$ there exists an ordinal $\alpha(k) < \omega_1$ for which $k \in \overline{\cup\{P(X_\alpha) : \alpha < \alpha(k)\}}$. Put $\alpha^* = max\{\alpha(k) : k \in K^*\}$. Then $K^* \subset \overline{\cup\{P(X_\alpha) : \alpha < \alpha^*\}}$. But $P(X_\alpha) = P(\cup\{X_\gamma : \gamma \leq \alpha\}) = M_{\alpha+1} \subset X_{\alpha+1}$, which implies $K^* \subset \overline{\cup\{X_{\alpha+1} : \alpha < \alpha^*\}} \subset \overline{X_{\alpha^*+1}} \subset P(X_{\alpha^*+1}) \subset M_{\alpha^*+2}$ and $x_{K^*} \in M_{\alpha^*+3} \subset X^*$, which contradicts n-pseudocompactness of P^*.

The same proof works for

THEOREM 1.2. *n-starcompactness is τ-extendable for any uncountable cardinal τ and every $n \in \omega$.*

COROLLARY 1.1. *Compactness, countably compactness and pseudocompactness are τ-extendable.*

PROOF. Compactness can be considered as 0-starcompactness, countably compact is equivalent with 1-pseudocompact, and pseudocompact - with 2-starcompact.

2. Berner's example $\chi = \mathcal{N} \cup \mathcal{S}$

Here we describe a slightly modified construction from [Ber]. Thus, we take arbitrary cardinal $\tau \geq @^+$ instead of $@^+$ in [Ber], and restrict the freedom of choice of certain "arbitrary family". Denote $D(\tau)$ the discrete space of power $\tau \geq @^+$, $\mathcal{C} = D^\omega$ the Cantor set, and $\mathcal{N} = \mathcal{C} \times D(\tau)$ the cartesian product. Choose any maximal (with respect to inclusion) almost disjoint subfamily $\mathcal{D}_0$ of the family of all countable subsets of $D(\tau)$.

Denote $\mathcal{S}_0$ the family of all sequences of $s = (U_n \times \{\alpha_n\} : n \in \omega)$ kind where U_n is a nonempty clopen subset of $\mathcal{C}$ and $\alpha_n \in D(\tau)$ (we consider $U_n \times \{\alpha_n\}$ as a subset of $\mathcal{C} \times D(\tau) = \mathcal{N}$) such that:

(1) $n \neq m \Rightarrow \alpha_n \neq \alpha_m$,
(2) $n \neq m \Rightarrow U_n \cap U_m = \emptyset$,
(3) the set $A(s) = \{\alpha_n : n \in \omega\}$ is an element of $\mathcal{D}_0$.

Let us say that a subfamily $\mathcal{T} \subset \mathcal{S}_0$ is *refined* if the following condition holds:

$$(*) \qquad \begin{aligned} &s = (U_n \times \{\alpha_n\} : n \in \omega) \in \mathcal{T} \ni s' = (V_m \times \{\beta_m\} : m \in \omega) \\ &\Rightarrow \text{ the set } I_{ss'} = \{(n,m) \in \omega \times \omega : \alpha_n = \beta_m \text{ and } U_n \cap V_m \neq \emptyset\} \text{ is finite.} \end{aligned}$$

The Berner's space χ is $\mathcal{N} \cup \mathcal{S}$, where $\mathcal{S}$ is a maximal (with respect to inclusion) refined subfamily of $\mathcal{S}_0$.

The topology on χ is defined as follows: $\mathcal{N}$ with its product topology is an open subspace while a basic neighbourhood of the point $s = (U_n \times \{\alpha_n\} : n \in \omega)$ is $O_m(s) = \{s\} \cup \cup \{U_n \times \{\alpha_n\} : n \geq m\}$ where $m \in \omega$.

It has been shown in [Ber] that χ is pseudocompact but is not countably pracompact.

3. G_δ-diagonal for χ

Generally, χ does not have a G_δ-diagonal, but with a special choice of $\mathcal{S}$ it does.

Let us consider the Cantor set $\mathcal{C}$ as a subspace of the unit interval of the real line with its usual metric; $diam(A)$ denotes the diameter of the set $A \subset \mathbb{R}$.

We will say that a sequence $s = (U_n \times \{\alpha_n\} : n \in \omega)$ is quick if for every $n \in \omega$

$$(**) \qquad diam(\cup\{U_m : m \geq n\}) < 1/2^n.$$

LEMMA 3.1. *For every sequence $s = (U_n \times \{\alpha_n\} : n \in \omega)$ there exists an increasing sequence of integers $(n(m) : m \in \omega)$ and a quick sequence $t = (V_m \times \{\beta_m\} : m \in \omega)$ such that $V_m \subset U_{n(m)}$ and $\alpha_{n(m)} = \beta_m$ for every $m \in \omega$.*

PROOF. Choose a point $x_n \in U_n$ for every $m \in \omega$. The sequence $\lambda = (x_n : n \in \omega)$ has at least one limit point $x^* \in \mathcal{C}_0$. Then there exist a subsequence $(x_{n(m)} : m \in \omega)$ of λ such that $|x_{n(m)} - x^*| < 1/2^{m+2}$. Put $W_m = B\left(x_{n(m)}, 1/2^{n(m)+2}\right) \cap U_{n(m)}$ where $B(x,\epsilon)$ is a ϵ-ball around x in $\mathcal{C}$, and $\beta_m = \alpha_{n(m)}$ for each $m \in \omega$, and take any clopen neighbourhood V_m such that $x_{n(m)} \in V_m \subset W_m$.

To show that $t = (V_m \times \{\beta_m\} : m \in \omega)$ is quick let us suppose that $y_1, y_2 \in \cup\{V_m : m \geq n\}$ for some $n \in \omega$. Then $y_i \in V_{m_i}$ for $i = 1,2$ and some $m_1, m_2 \geq n$, and we have $|y_1 - y_2| \leq |y_1 - x_{n(m_1)}| + |x_{n(m_1)} - x^*| + |x^* - x_{n(m_2)}| + |x_{n(m_2)} - y_2| \leq 1/2^{m_1+2} + 1/2^{m_1+2} + 1/2^{m_2+2} + 1/2^{m_2+2} \leq 1/2^{n+2} + 1/2^{n+2} + 1/2^{n+2} + 1/2^{n+2} = 1/2^n$.

PROPOSITION 3.1. *The family $\mathcal{S}$ can be chosen to consist of the quick sequences.*

PROOF. Let $\mathcal{S}_q \subset \mathcal{S}_0$ be the family of all quick sequences. It follows from Lemma 3.1 that a maximal refined subfamily of $\mathcal{S}_q$ is also a maximal refined subfamily of $\mathcal{S}_0$.

PROPOSITION 3.2. *If $\mathcal{S}$ consist of quick sequences, then $\chi = \mathcal{N} \cup \mathcal{S}$ has a G_δ-diagonal.*

PROOF. We shall construct a sequence $(\gamma_n : n \in \omega)$ of open covers of χ such that

$$(***) \qquad \cap\{St(x,\gamma_n) : n \in \omega\} = \{x\} \text{ for each } x \in X.$$

A space has such a sequence if and only if it has a G_δ-diagonal [Eng]. Put $\gamma_n = \gamma_{n,1} \cup \gamma_{n,2}$ where $\gamma_{n,1} = \{U \times \{\alpha\} : U \text{ is clopen in } \mathcal{C}, diam(U) < 1/2^n, \alpha \in D(\tau)\}$, and $\gamma_{n,2} = \{O_n(s) : s \in \mathcal{S}\}$.

To check (***), let us note that if $x \in \mathcal{S}$ then $St(x,\gamma_n) = O_n(x)$, and $\cap\{St(x,\gamma_n) : n \in \omega\} = \cap\{O_n(x) : x \in \omega\} = \{x\}$. Now, let $x = (x_0, \alpha) \in \mathcal{N}$. Suppose $y \in \cap\{St(x,\gamma_n) : n \in \omega\}$. Then $y \in \mathcal{S}$ is impossible by the above remark since $y \in St(x,\gamma)$ iff $x \in St(y,\gamma)$. So, we have $y = (y_0, \beta) \in \mathcal{N}$. Then $y_0 = x_0$ for otherwise $y \notin St(x,\gamma_n)$ for $n > -log_2|x_0 - y_0|$. Finally, $\alpha = \beta$ since for any $n \in \omega$ and any $W \in \gamma_n$ $(x,\alpha) \in W \ni (x,\beta) \Rightarrow (\alpha = \beta)$. So, $x = y$ which completes the proof.

4. Making χ 2-pseudocompact

Let $A \in \mathcal{D}_0$. We will call a sequence $\xi = ((x_n, \alpha_n) : n \in \omega)$ of points of $\mathcal{N}$ *A-fine* if the following holds:
(1) $x_n \neq x_m$ for $n \neq m$,
(2) $\alpha_n \neq \alpha_m$ for $n \neq m$,
(3) $\{\alpha_n : n \in \omega\} \subset A$,
(4) $|x_n - x_m| \leq 1/2^{min(n,m)+1}$ for any $n, m \in \omega$.

A sequence is *fine* if it is A-fine for some $A \in \mathcal{D}_0$. The families of all A-fine sequences and of all fine sequences we will denote $\mathcal{F}_A$ and $\mathcal{F}$, respectively.

For $A \in \mathcal{D}_0$ denote $S(A) = \{s \in \mathcal{S}_0 : A(s) = A\}$. Obviously, $|S(A)| = ©$, and $|\mathcal{F}_A| = ©$.

Henceforth, assume CH.

Let us order $\mathcal{F}_A$ on ω_1 type: $\mathcal{F}_A = \{\xi_\lambda : \lambda < \omega_1\}$. We will inductively define an increasing chain $\{S_\lambda(A) : \lambda < \omega_1\}$ of subfamilies of $S(A)$ such that for each $\lambda < \omega_1$ the following conditions hold:
(i) $\mathcal{T} = S_\lambda(A)$ satisfies (*),
(ii) for $\xi_\lambda = ((x_n, \alpha_n) : n \in \omega)$ there is a sequence $s = (U_m \times \{\beta_m\} : m \in \omega) \in S_\lambda(A)$ and increasing sequences of integers $\{n_i : i \in \omega\}$ and $\{m_i : i \in \omega\}$ such that $\alpha_{n_i} = \beta_{m_i}$, and $x_{n_i} \in U_{m_i}$ for each $i \in \omega$,
(iii) $S_\lambda(A)$ is countable.

Put $S_0(A) = \emptyset$.

Suppose $0 < \lambda_0 < \omega_1$ and the families $S_{\lambda'}(A)$ have been constructed for all $\lambda' < \lambda$ with (i), (ii), (iii) satisfied. Put $S^0_{\lambda_0}(A) = \cup\{S_{\lambda'}(A) : \lambda' < \lambda_0\}$. If (ii) holds for $\lambda = \lambda_0$ and $S^0_{\lambda_0}(A)$ as $S_\lambda(A)$ then put $S_{\lambda_0}(A) = S^0_{\lambda_0}(A)$. If not, let us order the countable family $S^0_{\lambda_0}(A)$ on type ω : $S^0_{\lambda_0}(A) = \{s_k : k \in \omega\}$; $s_k = \{U_{m(k)} \times \{\beta_m(k)\} : m \in \omega\}$ for each $k \in \omega$. We have $\xi_{\lambda_0} = ((x^0_n, \alpha_n) : n \in \omega)$. For each $n \in \omega$ denote $F_n = \cup\{U_m(k) : k \leq n, \beta_m(k) = \alpha_n, \text{and } x^0_n \notin U_m(k)\}$. Then F_n is clopen as a finite union of clopen sets, and $x^0_n \notin F_n$. There is, however, a clopen neighbourhood V_n of x^0_n such that $x^0_n \in V_n \subset B(x^0_n, 1/2^{n+2})$. Put $s_{\lambda_0} = (V_n \times \{\alpha_n\} : n \in \omega)$, and $S_{\lambda_0}(A) = S^0_{\lambda_0}(A) \cup \{s_{\lambda_0}\}$.

We shall check that $S_{\lambda_0}(A)$ satisfies (i), (ii), (iii). If $s, s' \in S_{\lambda_0}(A)$ then either they both belong to $S^0_{\lambda_0}(A)$, and hence to $S_{\lambda'}(A)$ for some $\lambda' < \lambda_0$, or one of them, say s', is equal s_{λ_0}. In the first case (*) holds by the inductive supposition. Consider the second case. We have $s = s_k = (U_m(k) \times \{\beta_n(k)\} : m \in \omega)$ for some $k < \omega$. Since (ii) did not hold for $\lambda = \lambda_0$, and $S^0_{\lambda_0}(A)$ as $S_\lambda(A)$, there is a number $l < \omega$ such that if $n, m > l$ then $\alpha_n \neq \beta_m$, or $x^0_n \notin U_m$. Put $j = max(k, l)$. Then for each $n, m > j$ we have $\alpha_n \neq \beta_m$, or $V_n \cap U_m = \emptyset$ by the definition of V_n.

(ii) and (iii) follow directly from the construction.

To prove that s_{λ_0} is a quick sequence, let us suppose that $y_1, y_2 \in \cup\{V_m : m \geq n\}$ for some $n \in \omega$. Then $y_i \in V_{m_i}$ for $i = 1, 2$ and some $m_1, m_2 \geq n$, and we have $|y_1 - y_2| \leq |y_1 - x_{n(m_1)}| + |x_{n(m_1)} - x^*| + |x^* - x_{n(m_2)}| + |x_{n(m_2)} - y_2| \leq 1/2^{m_1+2} + 1/2^{m_1+2} + 1/2^{m_2+2} + 1/2^{m_2+2} \leq 1/2^{n+2} + 1/2^{n+2} + 1/2^{n+2} + 1/2^{n+2} = 1/2^n$.

Finally, put $S_{\omega_1}(A) = \cup\{S_\lambda(A) : \lambda < \omega_1\}$. Obviously, $S_{\omega_1}(A)$ consist of quick sequences.

Claim 1. *$S_{\omega_1}(A)$ is a maximal subfamily of $S(A)$ with respect to (*).*

Proof. $S_{\omega_1}(A)$ satisfies (*) because so does $S_\lambda(A)$ for each $\lambda < \omega_1$, and any two elements of $S_{\omega_1}(A)$ belong to $S_\lambda(A)$ for some $\lambda < \omega_1$. If $s_0 = (W_n \times \{\alpha_n\} : n \in \omega) \in S(A) \setminus S_{\omega_1}(A)$ is a quick sequence, then, choosing $x_n \in W_n$ for each $n \in \omega$ we obtain an A-fine sequence $\xi = ((x_n, \alpha_n) : n \in \omega)$ which coinside with ξ_λ for some $\lambda < \omega_1$. Then the pair s_0, s where s is from (ii), does not satisfy (*) and thus s_0 can not be added to $S_{\omega_1}(A)$ without loosing (*).

Put $\mathcal{S} = \cup\{S_{\omega_1}(A) : A \in \mathcal{D}_0\}$.

Claim 2. *$\mathcal{S}$ satisfies (*).*

Indeed, if $S, S' \in \mathcal{S}$ then either they are elements of the same $S_{\omega_1}(A)$ and satisfy (*) by properties of $S_{\omega_1}(A)$, or $S \in S_{\omega_1}(A)$, and $S' \in S_{\omega_1}(A')$ with $A = A'$, then $A \cap A'$ is finite, and (*) holds again.

Claim 3. *$\mathcal{S}$ is a maximal subfamily of $\mathcal{S}_0$ with respect to (*).*

Suppose $s_0 = ((U_n, \alpha_n) : n \in \omega) \in \mathcal{S}_0 \setminus \mathcal{S}$. Then $A(s_0) = \{\alpha_n : n \in \omega\} \in \mathcal{D}_0$, and since $S_{\omega_1}(A(s_0))$ is maximal there is an element $s \in S_{\omega_1}(A(s_0))$ such that the pair (s_0, s) does not satisfy (*), and thus s_0 can not be added to $\mathcal{S}$.

So, our version of $\chi = \mathcal{N} \cup \mathcal{S}$ is pseudocompact, and has a G_δ-diagonal since $\mathcal{S}$ consists of quick sequences. Now, we will show that our χ is 2-pseudocompact. By Theorem 1.1, it suffies to prove that it satisfies the property $\mathcal{R}_@(2pc)$ where $2pc$ is 2-pseudocompactness. In fact, we will prove that χ satisfies $\mathcal{R}_@(cp)$ where cp is countable pracompactness.

Suppose $M \subset \chi, |M| \leq @$. Denote $M' = M \cup \cup\{O_1(x) : x \in M \cap S\}$. Then $|M'| \leq @$. Put $D(M) = \{d \in D(\tau) : \exists x \in C \text{ with } (x, d) \in M'\}$, and $P(M) = \overline{C \times D(M)}$.

We shall prove that so defined operation $M \to P(M)$ satisfies the properties (1) - (4) from Definition 1.2.

(1) is obvious: if $M_1 \subset M_2$ then $M_1' \subset M_2'$, $D(M_1) \subset D(M_2)$ and $P(M_1) \subset P(M_2)$.

The family $\{P(M) : M \in \chi, |M| \leq @\}$ is ω-conservative since χ is first countable.

To check (3), suppose $\mathcal{A}$ is a family of subsets of χ, $|\mathcal{A}| \leq \tau$ and $|M| \leq \tau$ for any $M \in \mathcal{A}$. Then $|\cup \mathcal{A}| \leq \tau$ and

$$
\begin{aligned}
P(\cup\mathcal{A}) &= \overline{\mathcal{C} \times D(\cup\mathcal{A})} = \overline{\mathcal{C} \times \{d \in D(\tau) : \exists x \in \mathcal{C} \text{ with } (x, d) \in (\cup\mathcal{A})'\}} \\
&= \overline{\mathcal{C} \times \{d \in D(\tau) : \exists x \in \mathcal{C} \text{ with } (x, d) \in \cup\{M' : M \in \mathcal{A}\}\}} \\
&= \overline{\mathcal{C} \times (\cup\{\{d \in D(\tau) : \exists x \in \mathcal{C} \text{ with } (x, d) \in M'\} : M \in \mathcal{A}\})} \\
&= \overline{\mathcal{C} \times \cup\{D(M) : M \in \mathcal{A}\}} = \overline{\cup\{\mathcal{C} \times D(M) : M \in \mathcal{A}\}} \\
&= \overline{\{\cup\overline{\mathcal{C} \times D(M)} : M \in \mathcal{A}\}} = \overline{\cup\{P(M) : M \in \mathcal{A}\}}.
\end{aligned}
$$

Finally, the next claim provides (4):

CLAIM 4. *If $L \in D(\tau), |L| \leq @$, then $|\overline{\mathcal{C} \times L}| \leq @$.*

Indeed, $\overline{\mathcal{C} \times L} = (\mathcal{C} \times L) \cup \{s \in \mathcal{S} : O_i(s) \cap (\mathcal{C} \times L) \neq \emptyset$ for each $i \in \omega\} = (\mathcal{C} \times L) \cup \bigcup\{S_{\omega_1}(A) : A \in \mathcal{D}_0,\ A \cap L$ is infinite$\}$. We have $|\mathcal{C} \times L| = @$, $|S_{\omega_1}(A)| = @$ for each $A \in \mathcal{D}_0$, and $|\{A \in \mathcal{D}_0 : A \cap L$ is infinite $\}| \leq @$ since $A \cap L = A' \cap L$ implies $A = A'$ if $A, A' \in \mathcal{D}_0$ (and $A \cap L$ is infinite), and there is at most @ countable subsets of L. So, $|\overline{\mathcal{C} \times L}| = @ + @ \cdot @ = @$.

To conclude the proof of the 2-pseudocompactness of χ, we must show that $P(M)$ is countably pracompact for any $M \in \chi$ with $|M| \leq @$. It suffices, however, to prove the following:

CLAIM 5. *$\overline{\mathcal{C} \times L}$ is countably pracompact if $L \subset D(\tau), |L| \leq @$.*

PROOF. If L is finite then $\overline{\mathcal{C} \times L} = \mathcal{C} \times L$ is compact. Suppose it is infinite. $\mathcal{C}$ has @ pairwise disjoint dense subspaces [Pyt]. Choose such a subspace $\mathcal{C}_d$ for each $d \in L$, so that $\mathcal{C}_d \cap \mathcal{C}_{d'} = \emptyset$ for $d \neq d'$. Put $Y = \cup\{\mathcal{C}_d \times \{d\} : d \in L\}$. Then Y is dense in $\mathcal{C} \times L$, and hence in $\overline{\mathcal{C} \times L}$. Let us show that Y is countably compact in $\overline{\mathcal{C} \times L}$. Suppose $\xi = ((x_n, \alpha_n) : n \in \omega)$ be a sequence of points in Y. If the set $A(\xi) = \{d \in L : \alpha_n = d$ for some $n \in \omega\}$ is finite then ξ has an accumulation point in $\mathcal{C} \times \{d\}$ for some $d \in A(\xi)$. If $A(\xi)$ is infinite we can choose a subsequence $\eta = ((x_{n_i}, \alpha_{n_i}) : i \in \omega)$ with pairwise distinct $\alpha_{n_i} - s$. In this case, $x_{n_i} - s$ are distinct too, because they belong to the different $\mathcal{C}_d - s$. The sequence η has a fine subsequence γ which satisfies (ii) with some $s \in \mathcal{S}$. Then every neighbourhood of s in χ meets infinitely many elements of ξ.

By the way, we found that the weak countably compactness is not τ-extendable in contrast to countably compactness and pseudocompactness.

5. One more example

In the previous section we constructed a CH example of a 2-pseudocompact space which is not countably pracompact. Here we give a ZFC example of such a space. Unfortunately, it has no G_δ-diagonal (in fact, it is not even first countable).

We are going to construct a space $\xi = \mathcal{N} \cup \mathcal{D}$ where $\mathcal{N}$ is the same that in the space $\chi = \mathcal{N} \cup \mathcal{S}$ considered above: $\mathcal{N} = \mathcal{C} \times \mathcal{D}(\tau), \tau \geq @^+$. As in section 4, we denote by $\mathcal{F}$ the family of all fine sequences.

Choose a maximal subfamily $\mathcal{D}$ of $\mathcal{F}$ according to the following property:

(*v)
$$d = (x_n : n \in \omega), d' = (y_n : n \in \omega), d \neq d' \Rightarrow x_n \neq y_m$$
for all but finitely many pairs (n, m).

The topology of $\mathcal{N} \cup \mathcal{D}$ is defined as follows: $\mathcal{N}$ (with the product topology) is an open subspace of $\mathcal{N} \cup \mathcal{D}$. To define local bases in the points of $\mathcal{D}$, for every $d = ((a_n, \alpha_n) : n \in \omega) \in \mathcal{D}$ denote by $F(d)$ the set of all mappings $f : \omega \to \mathcal{T}(\mathcal{N})$ (where $\mathcal{T}(\mathcal{N})$ is the topology of $\mathcal{N}$) for which $f(n)$ is a clopen neighbourhood of (a_n, α_n) in $\mathcal{C} \times \{\alpha_n\}$ for every $n \in \omega$. However, for each $d \in \mathcal{D}$, $f \in F(d)$, and $n \in \omega$ put $\tilde{O}_{f,n}(d) = \{d\} \cup \cup\{f(m) : m \geq n\}$ and$O_{f,n}(d) = \tilde{O}_{f,n}(d) \cup \{d' \in \mathcal{D} : \forall f' \in F(d') \exists n' \in \omega : f(m) \cap f'(m) \neq \emptyset$ for $m \geq max(n, n')\}$.

Note that if $d' = ((a'_m, \alpha'_m) : m \in \omega) \in \mathcal{D}$ and $d \neq d'$ then $d' \in O_{f,n}(d) \Leftrightarrow \exists n' \geq n : (a'_m, \alpha'_m) \in f(m), \forall m \geq n'$ (this is a consequence of the fact that the sets $f(n)$ are clopen in $\mathcal{N}$). This implies that d' is in $O_{f,n}(d)$ together with its neighbourhood of $O_{f',n'}$ type and therefore the family of sets $\mathcal{T}(\mathcal{N}) \cup \{O_{f,n}(d) : d \in \mathcal{D}, f \in F(d), n \in \omega\}$ is a base for topology on $\mathcal{N} \cup \mathcal{D}$ and the sets $O_{f,n}(d)$ are clopen in this topology. The property (*v) implies that $\mathcal{N} \cup \mathcal{D}$ is a Hausdorff space:

if $d, d' \in \mathcal{D}, d = ((a_n, \alpha_n) : n \in \omega), d' = ((a'_n, \alpha'_n) : n \in \omega)$, then either the set $\{\alpha_n : n \in \omega\} \cap \{\alpha'_n : n \in \omega\}$ is finite and then $O_{f,m}(d) \cap O_{f',m}(d') = \emptyset$ for any $f \in F(d), f' \in F(d')$ and $m \geq max\{n : \exists n' : \alpha_n = \alpha'_n\} + max\{n' : \exists n : \alpha_n = \alpha'_{n'}\} + 1$, or $(\alpha_n : n \in \omega) = (\alpha'_n : n \in \omega)$ - then eventually $a_n \neq a_{n'}$, and we can choose $m \in \omega$, and - for every $n \geq m$ - disjoint clopen neighbourhood $V_n, V'_n \subset \mathcal{C} \times \{\alpha_n\}$ of a_n and a'_n; set $f(n) = V_n, f'(n') = V'_n$ for $n \geq m$ - then $O_{f,m}(d) \cap O_{f',m}(d') = \emptyset$.

The existence of a base, consisting of clopen sets together with T_2 implies that $\mathcal{N} \cup \mathcal{D}$ is a Tychonoff space.

LEMMA [Ber]. *Suppose $(O_i : i \in \omega)$ is a sequence of nonempty sets in $\mathcal{C}$. Then there exists a subsequence $(O_{i(j)} : j \in \omega)$ and nonempty open sets $V_j \subset O_{i(j)}$ such that $V_j \cap V_k = \emptyset$ for $j \neq k$.*

Let us show now that $\mathcal{N} \cup \mathcal{D}$ is pseudocompact. Otherwise, there exists a sequence $\xi = (O_i \times \{\alpha_i\} : i \in \omega)$ where O_i is clopen in $\mathcal{C}$ for every $i \in \omega$, and the family ξ is discrete in $\mathcal{N} \cup \mathcal{D}$. Using Lemma, choose a subsequence $(i(j) : j \in \omega)$, the open sets $V_j \subset O_{i(j)}$ and points $x_j \in V_j$ for which $V_j \cap V_k = \emptyset$ if $j \neq k$. Let us consider a sequence $\delta = (\{x_j\} \times \{\alpha_{i(j)}\} : j \in \omega)$ of points of $\mathcal{N}$. As $\mathcal{D}$ is maximal with respect to (*), there exists a sequence $d = (y_n : n \in \omega) \in \mathcal{D}$ for which $y_{n(k)} = x_{j(k)} \times \{\alpha_{i(j(k))}\}$ for some increasing sequences $(j(k) : k \in \omega)$ and $(n(k) : k \in \omega)$. This point d destroyes the discretness of ξ. A contradiction.

The proof of the fact that $\mathcal{N} \cup \mathcal{D}$ is not countably pracompact is similar to the corresponding proof from [Ber]: if Y is dense in $\mathcal{N} \cup \mathcal{D}$, then $Y \cap (\mathcal{C} \times \{\alpha\}) \neq \emptyset$ for every $\alpha \in D(@^+)$, and so $|Y| > @$. As $|\mathcal{C}| = @$, there exists an element $c \in \mathcal{C}$ for which the set $Y \cap (\{c\} \times D(@^+))$ is infinite. Let $d = ((C, \alpha_i) : i \in \omega)$ be a sequence of its elements consisting of pairwise distinct points. This sequence has no limit points in $\mathcal{N} \cup \mathcal{D}$: if $d' = ((x_n, \alpha'_n) : n \in \omega)$, then $C = x_n$ for at most one x_n - denote this n as n^*: for all $n > n^*$ choose clopen sets V_n such that $C \notin V_n \ni x_n$. Put $f(n) = V_n$ for all $n > n^*$. Then $O_{n^*+1,f}(d') \cap d = \emptyset$.

CLAIM. *$P(A)$ is countably pracompact if $|M| \leq @$.*

The proof is the same to that for Claim 5 from the previous section. This claim implies that $P(M)$ is 2-pseudocompact if $|M| \leq @$.

Now, suppose $A \subset \mathcal{N} \cup \mathcal{D}, |A| \leq @$. Denote $M(A) = \{\alpha \in D(@^+) : \exists x \in \mathcal{C}$ for which $(x, \alpha) \in A\} \cup \{\alpha \in \overline{D(C^+) : \exists \alpha = ((x_n, \alpha_n) : n \in \omega) \in D \cap A}$ for which $\alpha_n = \alpha$ for some $n \in \omega\}$ and $P(A) = \overline{\mathcal{C} \times M(A)}$.

Clearly, $|M(A)| \leq @$ and $|P(A)| \leq @$ if $|A| \leq @$, i.e. condition (4) from Definition 1.2 for $\tau = @$ holds. Condition (1) is also true.

Check (2): let S be a family of subsets of $\mathcal{N} \cup \mathcal{D}$, $|A| \leq \tau$ for every $A \in S$, and $x \in \overline{\cup\{P(A) : A \in S\}}$. If $x \in \mathcal{N}$ then $x \in P(A)$ for some $A \in S$ that provides the desired result. Suppose $x \in D$. Note that for every basic neighbourhood O of x the set $A = \{\alpha \in D(\tau) : (x, \alpha) \in O$ for some $x \in \mathcal{C}\}$ is countable, and A contains an infinite subset A_0 such that for every $\alpha \in A_0$ we may choose some $A_\alpha \in S$ for which $\mathcal{C} \times \{\alpha\} \subset P(A_\alpha)$ holds. Then $x \in \overline{\cup\{P(A_\alpha) : \alpha \in A_0\}}$.

Now, check (3): suppose $\mathcal{A}$ is a family of subsets of power $\leq @$ in $\mathcal{N} \cup \mathcal{D}$, and $|\mathcal{A}| \leq @$. Note, that $M(\cup\mathcal{A}) = \cup\{M(A) : A \in \mathcal{A}\}$, and so $P(\cup\mathcal{A}) = \overline{\mathcal{C} \times M(\cup\mathcal{A})} = \overline{\mathcal{C} \times (\cup\{M(A) : A \in \mathcal{A}\})} = \overline{\cup\{\mathcal{C} \times M(A) : A \in \mathcal{A}\}} = \overline{\cup\{P(A) : A \in \mathcal{A}\}}$.

Finally $\mathcal{N} \cup \mathcal{D}$ satisfies the $\mathcal{R}_@(2pc)$ condition. From Theorem 1.1 we get 2-pseudocompactness of $\mathcal{N} \cup \mathcal{D}$.

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