Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 6:2 (1992), 275–280

AN EMBEDDING THEOREM FOR A^{β}_{sr} SPACES

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(Received 12.05.1992.)

Abstract. *A characterization is given of those measures μ on U, the upper half-plane, such that the inclusion map from the mixed norm space A^{β}_{sr}, $0 < s, r, \beta < \infty$, to the space $L^{p,q}(\mu)$, $0 < p, q < \infty$, is continuous.*

1. Introduction

If s, r, $\beta > 0$, a function f analytic on the upper half-plane $U = \{ z = x + iy : x \in R,\ y > 0 \}$ is said to belong to the space A^{β}_{sr} if

$$\|f\|^{r}_{s,r,\beta} = \int_0^{\infty} y^{r\beta-1} M_s(y,f)^r \, dy < \infty ,$$

where

$$M_s(y,f) = \left(\int_{-\infty}^{\infty} |f(x+iy)|^s \, dx \right)^{1/s} .$$

Let l_{sr} denote the mixed norm space of all double sequences $a = \{a_{jk}\}$, $j, k \in Z$, for which

$$\|a\|^{r}_{s,r} = \sum_j \left(\sum_k |a_{jk}|^s \right)^{r/s} < \infty , \quad 0 < s, r < \infty .$$

If $0 < p, q < \infty$, define

$$p \circ q = \left(\frac{1}{q} - \frac{1}{p} \right)^{-1} , \qquad 0 < q < p < \infty ;$$
$$p \circ q = \infty , \qquad 0 < p \le q < \infty .$$

Let $0 < p < \infty$, $0 < q < \infty$ and μ be some positive finite Borel measure on U. In this note we will find conditions on μ that are equivalent to the estimate:

There is a constant C such that

$$\left(\sum_j \left(\sum_k \int_{Q_{jk}} |f(z)|^p \, d\mu(z) \right)^{q/p} \right)^{1/q} \le C \|f\|_{s,r,\beta} , \quad \text{for all } f \in A^{\beta}_{sr} ,$$

1991 Mathematics Subject Classification: Primary 30.

Supported by Grant 0401A of FNS through Math. Inst. SANU

where Q_{jk} are squares

$$Q_{jk} = \{\, z = x + iy : k2^j \le x < (k+1)2^j \,,\, 2^j \le y < 2^{j+1} \,\} \,,$$

j and k are integers.

More precisely, we will prove the following theorem

THEOREM. *Let $0 < p, q, s, r < \infty$. There is a constant C such that*

$$\Big(\sum_j \Big(\sum_k \int_{Q_{jk}} |f(z)|^p \, d\mu(z)\Big)^{q/p}\Big)^{1/q} \le C\|f\|_{s,r,\beta} \,, \quad \textit{for all } f \in A^\beta_{sr} \,,$$

if and only if

$$\Big\| \Big\{ 2^{-j(\beta+1/s)} \mu\,(Q_{jk})^{1/p} \Big\} \Big\|_{u,v} < \infty \,,$$

where $u = s \circ p$, $v = r \circ q$.

A special case $0 < p < s < \infty$, $0 < q < r < \infty$, of Theorem was proved in [2]. The remaining three cases may be proved by using slighthy modified argument used in [2]. In this note we give details in case $0 < p < s < \infty$, $0 < r \le q < \infty$.

2. Interpolating sequences

Let $\rho(z,w)$ denote the pseudo hyperbolic metric

$$\rho(z,w) = \left| \frac{z-w}{z-\overline{w}} \right| , \quad z, w \in U \,.$$

For $a \in U$ and $0 < \delta < 1$ let

$$D_\delta(a) = \{\, z \in U : \rho(z,a) < \delta \,\}$$

The following lemma is a simple consequence of the fact that $D_\delta(a)$ is a disc in the Euclidean metric with center $C = \operatorname{Re} a + i\dfrac{1+\delta^2}{1-\delta^2}\operatorname{Im} a$ and radius $R = \dfrac{2\delta}{1-\delta^2}\operatorname{Im} a$.

LEMMA 2.1. *For given $0 < \delta < 1$ there exist positive integers n_1, n_2, n_3, and n_4 depending only on δ, such that if $a \in Q_{jk}$ then $D_\delta(a) \subset Q^{(\delta)}_{jk}$, where $Q^{(\delta)}_{jk}$ is a rectangle in U*

$$Q^{(\delta)}_{jk} = \{\, z = x + iy : (k - n_1)\, 2^j \le x < (k + 1 + n_2)\, 2^j, 2^{j-n_3} \le y < 2^{j+1+n_4} \,\} \,.$$

A sequence $\{z_{jk}\}$, $j, k \in Z$ in U is said to be δ-separated if there exists a $\delta > 0$ such that if $(m,n) \ne (j,k)$ then $\rho\,(z_{mn}, z_{jk}) \ge \delta$.

A sequence $\{z_{jk}\}$ in U is called an interpolation sequence for A^{β}_{sr} if whenever $\{x_{jk}\} \in l_{sr}$, then there exists $f \in A^{\beta}_{sr}$ satisfying $f(z_{jk})(\operatorname{Im} z_{jk})^{\beta+1/s} = x_{jk}$, i.e., if the operator R defined by $Rf = \left\{f(z_{jk})(\operatorname{Im} z_{jk})^{\beta+1/s}\right\}$ is a bounded map of A^{β}_{sr} onto l_{sr}.

It follows from the open mapping theorem that a constant M may be associated with any given interpolation sequence $\{z_{jk}\}$ such that any $\{x_{jk}\}$ with $\|\{x_{jk}\}\|_{sr} \leq 1$ is the image under R of a function $f \in A^{\beta}_{sr}$ with $\|f\|_{s,r,\beta} \leq M$. This M will be referred to as the interpolation constant of $\{z_{jk}\}$.

THEOREM A ([1]). *Let n, m be positive integers and $m_0 \in \{0,1,2,\dots,n-1\}$. Suppose that $\{z_{m_0+jn,k}\}$, $j,k \in Z$, is a sequence in U which satisfies the following conditions:*

(i) $\quad 2^{m_0+(j+1)n-1} \leq \operatorname{Im} z_{m_0+jn,k} < 2^{m_0+(j+1)n}, \quad$ *for all* $\quad k \in Z$;

(ii) $\quad \|\operatorname{Re} z_{m_0+jn,k_1} - \operatorname{Re} z_{m_0+jn,k_2}\| \geq (m+1)2^{m_0+(j+1)n-1}, \quad$ *if* $k_1 \neq k_2$.

If n and m are large enough, then T^{β}_{sr} defined by

$$T^{\beta}_{sr}(f) = \{ f(z_{m_0+jn,k})(\operatorname{Im} z_{m_0+jn,k})^{\beta+1/s} \}, \quad f \in A^{\beta}_{sr},$$

is a continuous linear map A^{β}_{sr} onto l_{sr}. In fact, there is a continuous linear map V of l_{sr} into A^{β}_{sr} so that $T^{\beta}_{sr}V$ is the identity mapping on l_{sr}.

Throughaut the paper we use C to denote positive constants, depending on the particular parameters r, s, M, δ, η, β, ... concerned in the particular problem in which they appear, not necessarily the same on any two occurrences.

3. Proof of Theorem

The following two lemmas will be needed in the proof of Theorem.

LEMMA 3.1 ([2], [5]). *Let $\delta > 0$ and $s, r, \beta > 0$. If f is a holomorphic function in U, then the following statements are equivalent:*

(i) $$f \in A^{\beta}_{sr},$$

(ii) $$\left\{ 2^{j(\beta+1/s)} \sup_{z \in Q_{jk}} |f(z)| \right\} \in l_{sr},$$

(iii) $$\left\{ 2^{j(\beta+1/s)} \sup_{z \in Q^{(\delta)}_{jk}} |f(z)| \right\} \in l_{sr}.$$

LEMMA 3.2 ([3]). *Let $0 < p, q, r, s < \infty$. Then, for any $\{x_{jk}\} \in l_{uv}$, we have*

$$\|\{x_{jk}\}\|_{u,v} = \sup_{\|\{y_{jk}\}\|_{s,r}=1} \|\{x_{jk}y_{jk}\}\|_{p,q},$$

where $u = s \circ p$, $v = r \circ q$.

Proof of Theorem. Let $0 < p < s < \infty$ and $0 < r \le q < \infty$. Note that in this case $u = \dfrac{sp}{s-p}$ and $v = \infty$.

Let $\left\| \left\{ 2^{-j(\beta+1/s)} \mu(Q_{jk})^{1/p} \right\} \right\|_{u,v} < \infty$ and $f \in A^{\beta}_{sr}$. By using Hölder's inequality, with index s/p, we obtain

$$\left(\sum_j \left(\sum_k \int_{Q_{jk}} |f(z)|^p \, d\mu(z) \right)^{q/p} \right)^{1/q} \le \left(\sum_j \left(\sum_k \sup_{z \in Q_{jk}} |f(z)|^p \mu(Q_{jk}) \right)^{q/p} \right)^{1/q}$$

$$\le \left(\sum_j \left(\sum_k \sup_{z \in Q_{jk}} |f(z)|^s \right)^{q/s} \left(\sum_k \mu(Q_{jk})^{u/p} \right)^{q/u} \right)^{1/q}$$

$$\le \left\| \left\{ 2^{-j(\beta+1/s)} \mu(Q_{jk})^{1/p} \right\} \right\|_{u,\infty} \left(\sum_j \left(\sum_k 2^{j(\beta+1/s)s} \sup_{z \in Q_{jk}} |f(z)|^s \right)^{q/s} \right)^{1/q}.$$

Since $r/q \le 1$, we have

$$\left(\sum_j \left(\sum_k 2^{j(\beta+1/s)s} \sup_{z \in Q_{jk}} |f(z)|^s \right)^{q/s} \right)^{1/q}$$

$$\le \left(\sum_j \left(\sum_k 2^{j(\beta+1/s)s} \sup_{z \in Q_{jk}} |f(z)|^s \right)^{r/s} \right)^{1/r} \le C \|f\|_{s,r,\beta} \qquad \text{by Lemma 3.1.}$$

Thus,

$$\left(\sum_j \left(\sum_k \int_{Q_{jk}} |f(z)|^p \, d\mu(z) \right)^{q/p} \right)^{1/q} \le C \left\| \left\{ 2^{-j(\beta+1/s)} \mu(Q_{jk})^{1/p} \right\} \right\|_{u,\infty} \|f\|_{s,r,\beta}.$$

To prove necessity we fix n and m large enough, so that any sequence satisfying the conditions of Theorem A is an interpolating sequence for A^{β}_{sr}, and construct a δ-lattice, that is, a δ-separated sequence $\{w_{jk}\}$ such that discs $D_\delta(w_{jk})$ cover U. Withaut loss of generality we may suppose that $2^{n-1+j} \le \operatorname{Im} w_{jk} < 2^{n+j}$, $j, k \in Z$.

Split $\{w_{jk}\}$ into $n_0 mn$ interpolating sequences for A^{β}_{sr} as it was done in [2]. We may suppose that all are η-separated for some $\eta > 4\delta$. Let $\{a_{jk}\} = \{w_{m_0+jn, s_k}\}$, $0 \le m_0 \le n-1$, be one of them.

By Theorem A, any sequence $\{y_{jk}\}$ with $\|\{y_{jk}\}\|_{s,r} = 1$ is of the form $\left\{ f(a_{jk}) (\operatorname{Im} a_{jk})^{\beta+1/s} \right\}$ for some $f \in A^{\beta}_{sr}$ with $\|f\|_{s,r,\beta} \le M$, where M is an interpolation constant associated with $\{a_{jk}\}$.

Thus, by Lemma 3.2,

$$\text{(3.1)} \qquad \left\| \left\{ 2^{-jn(\beta+1/s)} \mu(D_\delta(a_{jk}))^{1/p} \right\} \right\|_{u,v} \le C \left\| \left\{ f(a_{jk}) \mu(D_\delta(a_{jk}))^{1/p} \right\} \right\|_{p,q},$$

for some $f \in A^{\beta}_{sr}$ with $\|f\|_{s,r,\beta} \le M$.
As in [2] we find that

$$\begin{aligned}
&\Big(\sum_j \Big(\sum_k \int_{D_\delta(a_{jk})} |f(z)|^p \, d\mu(z)\Big)^{q/p}\Big)^{1/q} \\
(3.2) \qquad &\ge \Big(\Big(\sum_j \Big(\sum_k \|f(a_{jk})\|^p \mu(Q_\delta(a_{jk}))\Big)^{q/p}\Big)^{1/q} \\
&- \delta\Big(\sum_j \Big(\sum_k \mu(Q_\delta(a_{jk}))\Big) \sup_{z \in G^{(\eta)}_{jk}} |f(z)|^p\Big)^{q/p}\Big)^{1/q}.
\end{aligned}$$

Here, G_{jk} denotes the square $Q_{m_0+(j+1)n-1,s_k}$ which contains the point $w_{m_0+jn,s_k} = a_{jk}$ and $G^{(\delta)}_{jk}$ is the rectangle associated with G_{jk} (Lemma 2.1).

Proceed as in the proof of sufficiency to conclude that

$$\begin{aligned}
&\Big(\sum_j \Big(\sum_k \mu(Q_\delta(a_{jk}))\Big) \sup_{z \in G^{(\eta)}_{jk}} |f(z)|^p\Big)^{q/p}\Big)^{1/q} \\
(3.3) \qquad &\le C \Big\| \{2^{-jn(\beta+1/s)} \mu(D_\delta(a_{jk}))^{1/p}\} \Big\|_{u,\infty} \Big\| \{2^{j(\beta+1/s)} \sup_{z \in G^{(\eta)}_{jk}} |f(z)|^p\} \Big\|_{p,q} \\
&\le C \Big\| \{2^{-jn(\beta+1/s)} \mu(D_\delta(a_{jk}))^{1/p}\} \Big\|_{u,\infty} \|f\|_{s,r,\beta}, \quad \text{by Lemma 3.1.}
\end{aligned}$$

Since the discs $D_\delta(a_{jk})$ are disjoint, we have

$$\begin{aligned}
&\Big(\sum_j \Big(\sum_k \int_{D_\delta(a_{jk})} |f(z)|^p \, d\mu(z)\Big)^{q/p}\Big)^{1/q} \\
(3.4) \qquad &\le \Big(\sum_j \Big(\sum_k \int_{Q^{(\delta)}_{jk}} |f(z)|^p \, d\mu(z)\Big)^{q/p}\Big)^{1/q} \\
&\le C\Big(\sum_j \Big(\sum_k \int_{Q_{jk}} |f(z)|^p \, d\mu(z)\Big)^{q/p}\Big)^{1/q} \le C\|f\|_{s,r,\beta}
\end{aligned}$$

by assumption.

If we choose δ small enough, then from (3.1), (3.2), (3.3) and (3.4) we see that

$$\Big\| \{2^{-jn(\beta+1/s)} \mu(D_\delta(a_{jk}))^{1/p}\} \Big\|_{u,\infty} < \infty.$$

From this we conclude that

$$\Big\| \{2^{-j(\beta+1/s)} \mu(Q_\delta(w_{jk}))^{1/p}\} \Big\|_{u,\infty} < \infty$$

and consequently, since $\{w_{jk}\}$ is a δ-lattice,

$$\left\| \{ 2^{-j(\beta+1/s)} \mu (Q_{jk})^{1/p} \} \right\|_{u,\infty} < \infty .$$

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