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SOME PROPERTIES OF LOCALLY FINITE HYPERSPACE TOPOLOGY

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Abstract. *For a space X, let $\exp_f X$, $\exp_{lf} X$, $\exp_{clf} X$ be the collection of all nonempty closed subsets of X with the finite, locally finite, countable locally finite topology, respectively. Some separations properties of the space $\exp_{lf} X$ are investigated. If $\exp_f X$ is a real-compact space, then $\exp_{clf} X$ is a real-compact space. If X is a Lindelöf space, then $\exp_{lf} X$ is a real-compact space. A simple example show that there exists a non Lindelöf space X such that $\exp_{lf} X$ is a real-compact space.*

1. Preliminaries

Throughout this paper all spaces are assumed to be at least Hausdorff. The closure of a set $A \subset (X, \tau)$ is denoted by $[A]_X$, $[A]_\tau$ or $[A]$.

Let (X, τ) be a space. Then $\exp X$ denotes the collection of all nonempty closed subsets of X and $\mathcal{Z}(X)$ the collection of all nonempty compact subsets of X . If $\mathcal{U} = \{U_s, s \in S\}$ is a family of subsets of X we write

$$\langle \mathcal{U} \rangle = \langle U_s, s \in S \rangle = \{F \in \exp(X) : F \subset \cup U_s \text{ and } F \cap U_s \neq \emptyset, s \in S\}$$

The *finite, locally finite, countable locally finite topology* $\tau_{fin}, \tau_{lfin}, \tau_{clfin}$ on $\exp X$ is constructed taking as a base the sets of the form $\langle \mathcal{U} \rangle$, where $\mathcal{U}$ is a finite, locally finite, countable locally finite family of open subsets of X ([2], [4], [8], [9]). We denote the hyperspace $(\exp X, \tau_{fin}), (\exp X, \tau_{lfin}), (\exp X, \tau_{clfin})$ by $\exp_f X$, $\exp_{lf} X$, $\exp_{clf} X$, respectively. If $\tau_{fin} = \tau_{lfin}$ we denote $\exp_f X$ and $\exp_{lf} X$ by $\exp X$. Since ($\mathcal{Z}(X), \tau_{fin}) = (\mathcal{Z}(X), \tau_{lfin})$ we adopt the simplified notation $\mathcal{Z}(X)$.

Since $\tau_{fin} \subseteq \tau_{lfin}$, then the identity mapping $Id : \exp_{lf} X \longrightarrow exp_f X$ is a continuous mapping.

Note that (see [10])

(*a*) $\tau_{fin} = \tau_{lfin}$ iff each locally finite family of open subsets of X is finite.

(*b*) $\tau_{fin} = \tau_{lfin}$ iff $[\mathcal{Z}(X)]_{lfin} = \exp\ X$.

Lemma 1.1 ([10]). *Let $\mathcal{U} = \{U_s : s \in S\}$ and $\mathcal{V} = \{V_t : t \in T\}$ be two locally finite collections of non empty open subset of X. Then $\langle \mathcal{U} \rangle \subseteq \langle \mathcal{V} \rangle$ iff $\cup \mathcal{U} \subseteq \cup \mathcal{U}$ and each member of $\mathcal{V}$ contains a member of $\mathcal{U}$.*

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LEMMA 1.2. *Let* $\mathcal{U} = \{U_s : s \in S\}$ *and* $\mathcal{V} = \{V_s : s \in S\}$ *be two locally finite collections of non empty open subsets of* X*, then:*
(a) $[\langle U_s : s \in S\rangle]_{\tau_{lfin}} = \langle [U_s]_X : s \in S\rangle$.
(b) *If* $[V_s]_X \subseteq U_s \ \forall s \in S$*, then* $[\langle\mathcal{V}\rangle]_{lfin} \subseteq \langle\mathcal{U}\rangle$.

2. Separation properties

Our next theorems deals with separation properties of $\exp_{lf} X$ and $\exp_f X$.

THEOREM 2.1. *The following conditions are equivalent:*
(a) X *is regular.*
(b) $\exp_f X$ *is Hausdorff.*
(c) $\exp_{lf} X$ *is Hausdorff.*
PROOF. For (a) $\iff$ (b) see [4], [9] and for (a) $\iff$ (c) see [10].

THEOREM 2.2. *The following conditions are equivalent:*
(a) X *is normal.*
(b) $\exp_f X$ *is regular.*
(c) $\exp_f X$ *is completely regular.*
(d) $\exp_{lf} X$ *is regular.*
(e) $\exp_{lf} X$ *is completely regular.*

PROOF. The proof of (a) $\iff$ (b) $\iff$ (c) is given, for example in [4], [9].

The proof of the equivalence (a) $\iff$ (d) $\iff$ (e) is a modification of the proof of Theorem 3.8. in [4].

(a) $\implies$ (e). Let (X, τ) be a normal space, $\mathcal{U} = \{U_s : s \in S\}$ a locally finite family and $F_0 \in \langle\mathcal{U}\rangle$. The family $\{U_s, X \setminus F_0, s \in S\}$ is a open covering of X. Then there exists a closed covering $\{F_s, F_p : s \in S,\ p \neq s\}$ of X, $F_s \subset U_s$, $F_p \subset X \setminus F_0$. Suppose that $F_s \cap F_0 \neq \emptyset, \forall s \in S$ (if not, set $F_s' = F_s \cup \{x_s\}, x_s \in U_s \cap F_0$). By Urysohn lemma ([3], p. 75.), for $s \in S$ there exists $f_s : X \longrightarrow [0,1]$ such that $f_s(F_s) = 0$ and $f_s(X \setminus U_s) = 1$. For $r \in (0,1)$, let $V_s^r = \{x \in X : f_s(x) < r\}$. The family $\{V_s^r : s \in S\}$ is a locally finite family of open subsets of X, for each $r \in (0,1)$.

Let $\mathcal{W}_r = \langle V_s^r : s \in S\rangle$. If $q, r \in (0,1), q < r$, then $[V_s^q]_X \subset V_s^r$ and, by Lemma 1.2. it follows that $[\mathcal{W}_q]_{lf} \subset \mathcal{W}_r$.

Let $\mathcal{F}_1 = \exp X \setminus \langle\mathcal{U}\rangle$. Then the set $\mathcal{F}_1$ is a closed subset of $\exp_{lf} X$ and $F_0 \notin \mathcal{F}_1$ i.e. $\mathcal{F}_1 \cap \{F_0\} = \emptyset$. For $r, r' \in D, r < r'$ where the set D is dense in [0, 1], we have

$$\{F_0\} \subset \mathcal{W}_r \subset [\mathcal{W}_r]_{lf} \subset \mathcal{W}_{r'} \subset \exp X \setminus \mathcal{F}_1 = \langle\mathcal{U}\rangle.$$

By Theorem 2.17. in [4], there exists a mapping $\varphi : \exp_{lf} X \longrightarrow [0,1]$, such that $\varphi(F_0) = 0$ and $\varphi(\exp X \setminus \langle\mathcal{U}\rangle) = 1$. Thus the hyperspace $\exp_{lf} X$ is a completely regular space.

(e) $\implies$ (d). This is obvious.

(d) $\implies$ (a). Let $F_0, F_1 \in \exp X$ and $F_0 \cap F_1 = \emptyset$. The set $\mathcal{W} = \langle X \setminus F_1\rangle$ is an open set in $\exp_{lf} X$ and $F_0 \in \mathcal{W}$. Since $\exp_{lf} X$ is a regular space, then there exists a locally finite family $\{U_s : s \in S\}$ of open subsets of X such that $F_0 \in \langle U_s : s \in S\rangle$ and $[\langle U_s, s \in S\rangle]_{lf} \subset \mathcal{W}$. Therefore $F_0 \in U = \cup\{U_s, s \in S\}$, U is an open set in X

and $F_0 \subset U \subset X \setminus F_1$. If $[U]_X \cap F_1 = \emptyset$ i.e. $[U]_X \subset X \setminus F_1$, then X is a normal space.

Suppose that $[U]_X \cap F_1 \neq \emptyset$. Since $\{U_s : s \in S\}$ is a locally finite family, then $[\cup\{U_s : s \in S\}]_X = \cup\{[U_s]_X : s \in S\}$ and we have:

$$\cup\{([U_s] \cap F_1), s \in S\} = (\cup\{U_s : s \in S\}) \cap F_1 = [\cup\{U_s : s \in S\}]_X \cap F_1 \neq \emptyset.$$

There exists an index $t \in S$ such that $[U_t]_X \cap F_1 \neq \emptyset$. For every $s \in S$, let $x_s \in [U_s]_X, s \neq t$, and $x_t \in [U_t]_X \cap F_1$. The set $F = \{x_s : s \in S\}$ is a closed subset of X. Every neighbourhood of $F \in \exp_{lf} X$ contains some point of $\langle U_s : s \in S\rangle$ and so $F \in [\langle U_s : s \in S\rangle]_{lf} \subset \mathcal{W} = \langle X \setminus F_1\rangle$. This contradicts the fact that $F \cap F_1 \neq \emptyset$. Thus (d) $\Longrightarrow$ (a).

In [5] J. Keesling (see also N. Veličko [12]) has proved the following statement.

THEOREM 2.3. *The hyperspace* $\exp_f X$ *is a normal space if and only if* X *is a compact space.*

There exists a noncompact space X such that $\exp_{lf} X$ is a normal space. For example, the space $X = D(\aleph_0)$ ($D(\aleph_0)$ the discrete space of power $\aleph_0$), is a noncompact space, and $\exp_{lf} D(\aleph_0)$ is a normal space (the discrete space of power $2^{\aleph_0}$).

The following statement is valid.

THEOREM 2.4. *The following conditions are equivalent:*

(a) $\exp_{lf} X$ *is Lindelöf.*

(b) X *is compact* ($\exp_{lf} X$ *is compact*).

PROOF. The identity mapping $Id : \exp_{lf} X \longrightarrow \exp_f X$ is a continuous mapping. If $\exp_{lf} X$ is a Lindelöf space, $\exp_f X$ is also Lindelöf and hence a normal space. By Theorem 2.3. X is a compact space.

A space X is said to be *weakly Lindelöf* if every open covering $\{U_s : s \in S\}$ of X, has a countable subcovering $\{U_{s_i} : i \in N\}$ such that $\cup\{U_{s_i} : i \in N\}$ is a dense set in X (see [3]).

LEMMA 2.5. *Let* X *be a weakly Lindelöf space. Then every locally finite family of open sets in* X *is countable (see [3] p. 456).*

Lemma 2.5. implies

LEMMA 2.6. *Let* X *be a feebly Lindelöf space. Then*

(a) $\tau_{lf} = \tau_{clf}$,

(b) *every* $\langle \mathcal{U}\rangle \in \tau_{lf}$ *is a* G_δ*-set in* $\exp_f X$.

3. Real-compactness

Let X be a completely regular space, cX a compactification of X and $c : X \longrightarrow cX$ be a homeomorphic embedding of X into cX such that $[c(X)]_{cX} = cX$. In this section we shall identify the space X with the subspace $c(X) \subset cX$ of a compactification cX of X. The Stone-Čech compactification of X is denoted by βX.

In this section we will assume that X is normal so that $\exp_f X$, $\exp_{lf} X$ and $\exp_{clf} X$ are completely regular spaces. Hence, these three spaces have T_2 compactifications. The following statement was proved in [6].

LEMMA 3.1. *Let X be a normal space. Then $\exp\beta X$ is a compactification of the space $\exp_f X$.*

The mapping $j:\exp_f X \longrightarrow \exp\beta X$ defined by $j(F)=[F]_{\beta X}\ \forall F\in\exp_f X$ is a homeomorphic embedding.

A characterization of real-compactness is the following (see [1] p. 243.)

LEMMA 3.2. *A completely regular space Y is real-compact if and only if there exists a compactification bY of Y, such that for every point $y_0\in bY\setminus Y$ there exists a G_δ set G in bY such that $x_0\in G\subset bY\setminus Y$.*

For $\exp_f X$, Popov showed in [11] that if X is a Lindelöf space then the compactification $\exp\beta X$ of $\exp_f X$ satisfies the conditions of Lemma 3.2. Using this, Popov showed the following theorem:

THEOREM 3.3. *Let X be a Lindelöf space. Then the space $\exp_f X$ is real-compact.*

The following result is valid:

LEMMA 3.4. *If $\exp_f X$ is a real-compact space, then $\exp_{clf} X$ is also a real-compact space.*

PROOF. Let $\exp_f X$ be a real-compact space and let $b\exp_f X$ denote a compactification which satisfies the condition of Lemma 3.2. The mapping $j:\exp_{clf} X\longrightarrow \exp_f X\subset b\exp_f X$ given by: $\forall F\in\exp_{clf}X$, $j(F)=b(Id(F))=b(F)\in b(\exp_f X)$ $\subset b\exp_f X$ is a continuous one to one mapping. Then there exists a compactification $c\exp_{clf} X$ and an extension, $\tilde{\jmath}: c\exp_{clf} X\longrightarrow b\exp_f X$ of j (see [1] p. 334).

We are going to show that $c\exp_{clf} X$ also satisfies the condition of Lemma 3.2.

Let $T\in c\exp_{clf}X\setminus\exp_{clf}X$ and $\tilde{\jmath}(T)=\tilde{T}\in b\exp_f X$.

(i) $\tilde{\jmath}(T)=\tilde{T}\in b\exp_f X\setminus\exp_f X$. Then there exists a G_δ -set $\mathcal{G}\subset b\exp_f X$ such that $\tilde{T}\in\mathcal{G}\subset b\exp_f X\setminus\exp_f X$. Then we have that $\tilde{\jmath}^{-1}(\mathcal{G})$ is a G_δ -set, and

$$T\in\tilde{\jmath}^{-1}(\mathcal{G})\subset c\exp_{clf}X\setminus\exp_{clf}X.$$

(ii) $\tilde{\jmath}(T)=F_f\in\exp_f X$. Let $F_f=j(F_c)$, $F_c\in\exp_{clf}X$ ($F_f=F_c$ as elements of $\exp X$).

Let $\mathcal{U}(T)$ and $\mathcal{U}(F_c)$ be disjoint neighbourhoods of the points T and F_c in $c\exp_{clf}X$ and

$$\mathcal{U}(F_c)\cap\exp_{clf}X=\begin{cases}\langle U_1,U_2,\dots,U_k\rangle, & \text{if } F_c \text{ is countably compact },\\ \langle O_n:\ n\in N\rangle, & \text{if } F_c \text{ is not countably compact.}\end{cases}$$

Let now $\mathcal{V}(F_f)$ be a G_δ -set in $b\exp_f X$ such that $j(F_c)=F_f\in\mathcal{V}(F_f)$ and

$$\mathcal{V}(F_f)\cap\exp_f X=\begin{cases}\langle U_1,U_2,\dots,U_k\rangle, & \text{if } F_f \text{ is countably compact,}\\ \langle O_n:\ n\in N\rangle, & \text{if } F_f \text{ is not countably compact,}\end{cases}$$

$(\langle U_1,\dots,U_k\rangle=j(\langle U_1,\dots,U_k\rangle),\ \langle O_n:\ n\in N\rangle\ =\ j(\langle O_n:\ n\in N\rangle))$.

Suppose that for every G_δ - set $\mathcal{G}$, $T\in\mathcal{G}\subset\mathcal{U}(F_f)$, $\mathcal{G}\cap\exp_{clf}X\neq\emptyset$ holds. Then $\tilde{\jmath}(\mathcal{G})\not\subset\mathcal{V}(F_f)$, which contradicts the fact that $\tilde{\jmath}$ is a continuous mapping.

Hence, there exists a G_δ - set $\mathcal{G}_0$ such that $T \in \mathcal{G}_0 \subset c\exp_{clf} X \setminus \exp_{clf} X$, or $\tilde{\jmath}(T) \in b\exp_f X \setminus \exp_f X$.

The following statement (which can be also proved directly) is a consequence of Theorem 3.3. and Lemma 3.4.

THEOREM 3.5. *If X is a Lindelöf space, then the space $\exp_{lf} X$ is a real - compact space.*

There exists a space X which is not Lindelöf and $\exp_{lf} X$ is a real - compact space. Let $m > \aleph_0$ be a non-measurable cardinal number (for example $m = c$). Then the space $D(m)$ is not Lindelöf, and the space $\exp_{lf} D(m) = D(2^m)$ is real - compact.

REFERENCES

[1] A. V. ARKHANGEL'SKII, V. I. PONOMAREV, *Osnovy obshchei topologii v zadachah i uprazhneniyah*, NAUKA, Moskva, 1974.

[2] G. A. BEER, C. J. HIMMELBERG, K. PRIKRY AND F. VAN VLECK, *The locally finite topology on 2^X*, Proc. Amer. Math. Soc. 101(1987), 163-172.

[3] R. ENGELKING, *Obshchaya Topologiya*, MIR, Moskva, 1986.

[4] V. V. FEDORCHUK, V. V. FILIPOV, *Obshchaya Topologiya; osnovnye konstrukcii*, M. MGU, Moskva, 1988.

[5] J. KEESLING, *On the equivalence of normality and compactness in hyperspaces*, Pacific J. of Math. 33(1970), 657-667.

[6] J. KEESLING, *Normality and properties related to compactness in hyperspaces*, Proc. Amer. Math. Soc. 24(1970), 760-766.

[7] G. DI MAIO AND S. A. NAIMPALLY, *The locally finite hypertopology and generalized uniformities*, Zbornik Rad. Fil. Fak. (Niš) Ser. Mat. 5(1991), 109-112.

[8] M. MARJANOVIĆ, *Topologies on collections of closed subsets*, Publ. Inst. Math. (Beograd) 20(1966), 125-130.

[9] E. MICHAEL, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71(1951), 152-182.

[10] S. A. NAIMPALLY AND P. L. SHARMA, *Fine uniformity and the locally finite hyperspace topology*, Proc. Amer. Math. Soc. 103(1988), 641-646.

[11] V. V. POPOV, *O nekotoryh svojstvah eksponenty v topologii Vietorisa*, Top. Struktury i ih Otobrazh. LGU (Riga) (1987), 96-101.

[12] N. V. VELICHKO, *O prostransve zamknutyh mnozhestv*, Sib. Mat. Zh. 16 (1975), 627-629.

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