Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 6:2 (1992), 259–268

ON $\theta\delta$-CONTNUOUS FUNCTIONS

GRAZIA SANTORO

(Received 02. 03. 1992.)

ABSTRACT. *In this paper the $\theta\delta$-continuous functions are introduced and investigated as functions for which the inverse image of each θ-open set is a δ-open set. This completes the last open case in this topic.*

0. Introduction

In the "non regular" topology the weak forms of continuity have the same role that the continuity has in the classic topology. In literature there exist different weak forms of continuity (cfr [A], [C], [CL], [CN], [LH], [MR]) and some of them are characterized through δ-open and θ-open sets which are particular open sets introduced by Veličko [V].

Since these functions are characterized also providing that the inverse image of a "particular" open set is another "particular" open set, we put ourselves the problem, considering the family of θ-open subsets (τ_θ topology) and the family of δ-open subsets (τ^* topology), to test all possible cases of weak continuity which are obtained provided that the inverse image of a

1)	*θ-open set is*	*an open set,*	4)	*δ-open set is*	*an open set,*
2)	//	*a θ-open set,*	5)	//	*a θ-open set,*
3)	//	*a δ-open set,*	6)	//	*a δ-open set.*

We recall that *almost continuous* [SS], *δ-continuous* [C], *faintly continuous* [LH], *almost strongly θ-continuous* [NS] and *u-continuous* [G] functions may be characterized in the following way:

a function $f : X \to Y$ is

a) *almost continuous* $\iff$ the inverse image of each δ-open is open [C];
b) *δ-continuous* $\iff$ the inverse image of each δ-open set is δ-open [C];
c) *faintly continuous* $\iff$ the inverse image of each θ-open set is open [LH];
d) *almost strongly θ-continuous* $\iff$ the inverse image of each δ-open set is θ-open [MR];
e) *u-continuous* $\iff$ the inverse image of each θ-open set is θ-open [G].

1991 Mathematics Subject Classification: 54C10, 54C20.

Thiis research was supported by a grant from the C.N.R. (G.N.S.A.G.A.) and MURST through "Fondi 40%".

Then, among the six weak forms of continuity written above, we have to study only the 3) that we call *$\theta\delta$-continuous function.*

After some preliminaries given in section 1, in section 2 we define and characterize the $\theta\delta$-continuous functions, in section 3 we give some properties and implications of them with the other weak forms of continuity and, in section 4, we study the algebra of $\theta\delta$-continuous functions.

1. Preliminaries

Throughout the present paper X and Y always denote topological spaces, x an element of X, $\mathcal{U}_x$ the filter of neighbourhoods of x in X and $\mathcal{U}(\mathcal{F})$ the filter of neighbourhoods of the filter $\mathcal{F}$. Let $\mathcal{F}$ and $\mathcal{G}$ be filters. We write $\mathcal{F} \preceq \mathcal{G}$ iff for every $G \in \mathcal{G}$ there exists $F \in \mathcal{F}$ such that $F \subseteq G$.

If X is a topological space, and A a subset of X, then $\overline{A}$ or $\mathrm{cl}(A)$ and $\overset{\circ}{A}$ or $\mathrm{int}(A)$ denote the closure and interior of A, respectively.

The set of points $x \in X$ such that $\overline{U} \cap A \neq \emptyset$ ($\overset{\circ}{\overline{U}} \cap A \neq \emptyset$), for each neighbourhood $U \in \mathcal{U}_x$ of x, is called the *θ-closure* (resp. *δ-closure*) of A and it is denoted by $\overline{A}^\theta$ or $\mathrm{cl}_\theta(A)$ (resp. $\overline{A}^\delta$ or $\mathrm{cl}_\delta(A)$). The *θ-interior* (*δ-interior*) of A, denoted by $\overset{\circ\theta}{A}$ or $\mathrm{int}_\theta(A)$ (resp. $\overset{\circ\delta}{A}$ or $\mathrm{int}_\delta(A)$), is defined as the set of all $x \in A$ such that $\overline{U} \subset A$ (resp. $\overset{\circ}{\overline{U}} \subset A$) for some neighbourhood $U \in \mathcal{U}_x$ of x [V]. In general for $A \subset X$ we have the inclusions $A \subset \overline{A} \subset \overline{A}^\delta \subset \overline{A}^\theta$ and $A \supset \overset{\circ}{A} \supset \overset{\circ\delta}{A} \supset \overset{\circ\theta}{A}$. A subset $A \subset X$ is said to be *θ-closed* (*δ-closed*) if $A = \overline{A}^\theta$ (resp. $A = \overline{A}^\delta$); if $A = \overset{\circ\theta}{A}$ ($A = \overset{\circ\delta}{A}$) A is said to be *θ-open* (resp. *δ-open*). Note that the complement of a θ-open (δ-open) set is θ-closed (resp. δ-closed) and conversely. Note also that the θ-closure and δ-closure operators in general are not idempotent, i.e. in general $\overline{A}^\theta$ is not θ-closed and $\overline{A}^\delta$ is not δ-closed. This suggests to introduce the following

1.1. DEFINITION [BC]. *Let X be a topological space and A a subset of X. We define θ (resp. δ)-closed hull of A, denoted by $[A]_\theta$ (resp. $[A]_\delta$), as the smallest θ-closed (resp. δ-closed) subset of X containing A.*

Analogously we denote by $]A[_\theta$ (resp. $]A[_\delta$) the greatest θ-open (resp. δ-open) subset of X contained in A.

Note that for each subset $A \subset X$ we have $A \subset \overline{A} \subset [A]_\theta$, but if X is regular then $\overline{A} = \overline{A}^\theta = [A]_\theta$.

The topology determined by the δ-open (resp. θ-open) subsets of (X, τ) is denoted by τ^* (resp. τ_θ) and it is called *semiregularization*; its base is the family of regular open subsets $B = \overset{\circ}{\overline{B}}$. Obviusly in a topological space (X, τ) $\tau_\theta \subset \tau^* \subset \tau$ and X is regular if and only if $\tau_\theta \equiv \tau^* \equiv \tau$, X is *almost regular* [SA] if and only if $\tau^* \equiv \tau_\theta$ and X is *semiregular* if and only if $\tau \equiv \tau^*$.

2. Characterizations

2.1. DEFINITION. *Let X and Y be topological spaces. A function $f : X \to Y$ is said to be $\theta\delta$-continuous if the inverse image of each θ-open subset of Y is δ-open in X.*

From this definition we have

2.2. THEOREM. *A function $f : (X, \tau) \to (Y, \sigma)$ is $\theta\delta$-continuous if and only if $f : (X, \tau^*) \to (Y, \sigma_\theta)$ is continuous.*

2.3. COROLLARY. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function, where X is semiregular and Y is regular. Then f is $\theta\delta$-continuous if and only if it is continuous.*

2.4. THEOREM. *A function $f : (X, \tau) \to (Y, \sigma)$ is $\theta\delta$-continuous if and only if the inverse image of each θ-closed subset of Y is δ-closed in X.*

In order to give some characterizations of $\theta\delta$-continuous functions it is useful to put before some definitions and propositions.

2.5. DEFINITION. *A subset U of X is said to be a θ-neighbourhood of a point $x \in X$ if there exists a θ-open subset $A \subset X$ such that $x \in A \subset U$.*

Obviously the family $\theta\mathcal{U}_x$ of all θ-neighbourhoods of a point $x \in X$ is a filter in X. If $\mathcal{U}(\overline{\mathcal{U}}_x)$ denotes the filter of neighbourhoods of $\overline{\mathcal{U}}_x = \{\overline{U} : U \in \mathcal{U}_x\}$ we have the following

2.6. PROPOSITION. *Let (X, τ) be a topological space. A subset $A \subset X$ is θ-open if and only if $A \in \mathcal{U}(\overline{\mathcal{U}}_x)$ for each $x \in A$.*

PROOF. If A is a θ-open subset of X then, for each $x \in A$, there exists $U \in \mathcal{U}_x$ such that $\overline{U} \subset A$, hence $A \in \mathcal{U}(\overline{\mathcal{U}}_x)$. Conversely, if $x \in A$ and $A \in \mathcal{U}(\overline{\mathcal{U}}_x)$; then there exists $U \in \mathcal{U}_x$ such that $\overline{U} \subset A$, hence A is θ-open. $\square$

2.7. PROPOSITION. *For each $x \in X$, $\overline{\mathcal{U}}_x \preceq \mathcal{U}(\overline{\mathcal{U}}_x) \preceq \theta\mathcal{U}_x$.*

PROOF. The first inequality is obvious. We prove that $\mathcal{U}(\overline{\mathcal{U}}_x) \preceq \theta\mathcal{U}_x$. Let $U \in \theta\mathcal{U}_x$. Then there exists a θ-open subset A in X such that $x \in A \subset U$. From 2.6 it follows $A \in \mathcal{U}(\overline{\mathcal{U}}_x)$, and the proof is complete. $\square$

2.8. THEOREM. *A function $f : (X, \tau) \to (Y, \sigma)$ is $\theta\delta$-continuous if and only if for each $x \in X$ and and for each $V \in \theta\mathcal{U}_{f(x)}$, there exists $U \in \mathcal{U}_x$ such that $f(\overset{\circ}{\overline{U}}) \subset V$.*

PROOF. Assume $x \in X$ and $V \in \theta\mathcal{U}_{f(x)}$ a θ-neighbourhood of $f(x)$ in Y; there exists a θ-open subset $W \subset Y$ such that $f(x) \in W \subset V$. By hypothesis it follows that $f^{-1}(W)$ is δ-open and $x \in f^{-1}(W) \subset f^{-1}(V)$, hence there exists $U \in \mathcal{U}_x$ such that $x \in \overset{\circ}{\overline{U}} \subseteq f^{-1}(W) \subseteq f^{-1}(V)$. Therefore $f(x) \in f(\overset{\circ}{\overline{U}}) \subseteq ff^{-1}(W) \subseteq ff^{-1}(V) \subset V$. Conversely, let V be a θ-open subset of Y. For each $x \in f^{-1}(V)$, we have $V \in \theta\mathcal{U}_{f(x)}$. By hypothesis there exists $U \in \mathcal{U}_x$ such that $f(\overset{\circ}{\overline{U}}) \subset V$, hence $x \in \overset{\circ}{\overline{U}} \subseteq f^{-1}f(\overset{\circ}{\overline{U}}) \subseteq f^{-1}(V)$, i.e. $f^{-1}(V)$ is δ-open. $\square$

2.9. Corollary. *For a function $f : X \to Y$ the following are equivalent:*
i) *f is $\theta\delta$-continuous;*
ii) *for each $x \in X$ and for each θ-open subset V containing $f(x)$, there exists a regular open subset $U \subset X$ containing x such that $f(U) \subset V$;*
iii) *for each $x \in X$ and for each θ-open subset V containing $f(x)$, there exists a δ-open subset $U \subset X$ containing x such that $f(U) \subset V$.*

Proof. This follows from definitions and the fact that any δ-open set is a union of regular open subsets. □

2.10. Proposition. *A function $f : (X, \tau) \to (Y, \sigma)$ is $\theta\delta$-continuous if and only if for each filter $\mathcal{F} \preceq \overset{\circ}{\overline{\mathcal{U}}}_x$, $f(\mathcal{F}) \preceq \theta\mathcal{U}_{f(x)}$.*

Proof. Assume $x \in X$ and $V \in \theta\mathcal{U}_{f(x)}$. Since f is $\theta\delta$-continuous there exists $U \in \mathcal{U}_x$ such that $f(\overset{\circ}{\overline{U}}) \subseteq V$; moreover $\mathcal{F} \preceq \overset{\circ}{\overline{\mathcal{U}}}$ implies that there exists $F \in \mathcal{F}$ such that $F \subseteq \overset{\circ}{\overline{U}}$, hence $F \subseteq f(\overset{\circ}{\overline{U}}) \subseteq V$, i.e. $f(\mathcal{F}) \preceq \theta\mathcal{U}_{f(x)}$. The converse is obvious; from assumption $\mathcal{F} \equiv \overset{\circ}{\overline{\mathcal{U}}}$, hence $\overset{\circ}{\overline{\mathcal{U}}} \preceq \overset{\circ}{\overline{\mathcal{U}}}$ and $f(\overset{\circ}{\overline{\mathcal{U}}}) \preceq \theta\mathcal{U}_{f(x)}$, which means that f is $\theta\delta$-continuous. □

2.11. Theorem. *For a function $f : X \to Y$ the following are equivalent:*
i) *f is $\theta\delta$-continuous;*
ii) *$f(\overline{A}^{\delta}) \subset [f(A)]_\theta$ for each $A \subset X$;*
iii) *$\overline{f^{-1}(B)}^{\delta} \subset f^{-1}([B]_\theta)$ for each $B \subset Y$;*
iv) *$f^{-1}(]B[_\theta) \subset \mathrm{int}_\delta(f^{-1}(B))$ for each $B \subset Y$.*

Proof. i) $\Longrightarrow$ ii). Assume $x \in \overline{A}^{\delta}$, i.e. for each $U \in \mathcal{U}_x$, $\overset{\circ}{\overline{U}} \cap A \neq \emptyset$, and let $y = f(x)$. If $y \notin [f(A)]_\theta$ then $y \in Y \setminus [f(A)]_\theta$, where $Y \setminus [f(A)]_\theta$ is a θ-open set. Hence x belongs to the δ-open set $X \setminus f^{-1}([f(A)]_\theta)$. Thus there exists $U \in \mathcal{U}_x$ such that $\overset{\circ}{\overline{U}} \subset X \setminus f^{-1}([f(A)]_\theta)$, i.e. $\overset{\circ}{\overline{U}} \cap f^{-1}([f(A)]_\theta) = \emptyset$. But $A \subset f^{-1}f(A) \subset f^{-1}([f(A)]_\theta)$ and therefore $\overset{\circ}{\overline{U}} \cap A = \emptyset$, which is a contradiction.

ii) $\Longrightarrow$ iii). From ii) we have $f(\overline{f^{-1}(B)}^{\delta}) \subset [ff^{-1}(B)]_\theta \subset [B]_\theta$, and therefore $\overline{f^{-1}(B)}^{\delta} \subset f^{-1}([B]_\theta)$.

iii) $\Longrightarrow$ i). By Theorem 2.4 it is equivalent to prove that the inverse image of a θ-closed subset is δ-closed. Let A be a θ-closed subset of Y. Then $A = \overline{A} = [A]_\theta$ and, from iii), we have $\overline{f^{-1}(A)}^{\delta} \subset f^{-1}(A)$ and therefore $f^{-1}(A) = \overline{f^{-1}(A)}^{\delta}$,i.e. $f^{-1}(A)$ is δ-closed.

i) $\Longrightarrow$ iv). Assume $x \in f^{-1}(]B[_\theta)$. If $x \notin \mathrm{int}_\delta(f^{-1}(B))$ then $\overset{\circ}{\overline{U}} \cap (X \setminus f^{-1}(B)) \neq \emptyset$ for each $U \in \mathcal{U}_x$. By hypothesis $f^{-1}(]B[_\theta)$ is δ-open, so that there exists some $V \in \mathcal{U}_x$ such that $\overset{\circ}{\overline{V}} \subset f^{-1}(]B[_\theta) \subseteq f^{-1}(B)$. This is a contadiction.

iv) $\Longrightarrow$ i). Let A be a θ-open subset of Y. Then $A =]A[_\theta$ and $f^{-1}(A) \subset \mathrm{int}_\delta(f^{-1}(A))$. Since $f^{-1}(A) \supset \mathrm{int}_\delta(f^{-1}(A))$ it follows that $f^{-1}(A) = \mathrm{int}_\delta(f^{-1}(A))$, i.e. $f^{-1}(A)$ is δ-open. $\square$

2.12. THEOREM. *Let $f : X \to Y$ be $\theta\delta$-continuous function. Then for each subset $A \subset Y$, is $\mathrm{cl}_\delta\big(f^{-1}\big(\mathrm{int}(\overline{A}^\delta)\big)\big) \subseteq f^{-1}([A]_\theta)$.*

PROOF. Since $\overline{A}^\delta \subset \overline{A}^\theta \subset [A]_\theta$, one has $\mathrm{int}(\overline{A}^\delta) \subset \overline{A}^\delta \subset [A]_\theta$, and thus $f^{-1}\big(\mathrm{int}(\overline{A}^\delta)\big) \subset f^{-1}([A]_\theta)$ and $\mathrm{cl}_\delta\big(f^{-1}\big(\mathrm{int}(\overline{A}^\delta)\big)\big) \subset \mathrm{cl}_\delta\big(f^{-1}([A]_\theta)\big)$. As $[A]_\theta$ is θ-closed, from Theorem 2.4 we have that $f^{-1}([A]_\theta)$ is δ-closed. Therefore $\mathrm{cl}_\delta\big(f^{-1}\big(\mathrm{int}(\overline{A}^\delta)\big)\big) \subset \mathrm{cl}_\delta\big(f^{-1}([A]_\theta)\big) = f^{-1}([A]_\theta)$. $\square$

2.13. REMARK. For an open subset A of Y the condition of the previous theorem becomes: $\mathrm{cl}_\delta\big(f^{-1}(\mathrm{int}\,\overline{A})\big) \subseteq f^{-1}([A]_\theta)$. In fact, for each open A we have $\overline{A} = \overline{A}^\delta = \overline{A}^\theta$ ([V], Lemma 2).

3. Implications and properties

3.1. DEFINITION. *A function $f : X \to Y$ is said to be weakly θ-continuous* [CN_1] *(resp. θ-continuous* [AD]*, weakly continuous* [L]*) if for each $x \in X$ and each open neighbourhood V of $f(x)$, there exists an open neighbourhood U of x such that $f(\mathring{\overline{U}}) \subset \overline{V}$ (resp. $f(\overline{U}) \subset \overline{V}$, $f(U) \subset \overline{V}$).*

3.2. THEOREM. *Any weakly θ-continuous function is $\theta\delta$-continuous.*

PROOF. It follows from the following result: for each weakly θ-continuous function the inverse image of a θ-open subset is δ-open [CN_1]. $\square$

In general the converse is not true as the following simple example shows

3.3. EXAMPLE. Let $X = \{a, b, c\}$ be a set, and let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ be two topologies on X. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is $\theta\delta$-continuous but not weakly θ-continuous at the point c. $\square$

3.4. THEOREM. *Any continuous function is $\theta\delta$-continuous.*

PROOF. It follows from Theorem 3.2 and the fact that any continuous function is weakly θ-continuous [CN_1]. $\square$

3.5. THEOREM. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function, with (Y, σ) regular. Then f is $\theta\delta$-continuous if and only if it is continuous.*

PROOF. By Theorem 3.4 it is sufficient to prove that if f is $\theta\delta$-continuous and (Y, σ) is regular, then f is continuous. Let A an open subset of Y. Since Y is regular then $\sigma \equiv \sigma_\theta$, i.e. A is θ-open; by hypothesis $f^{-1}(A) \in \tau^*$. But $\tau^* \subset \tau$ and therefore the proof is complete. $\square$

Note that $\theta\delta$-continuous functions are not weakly continuous, in general, as it is shown by the following

3.6. EXAMPLE. Assume $X = \{0, 1\}$ endowed with the topology $\tau = \{\emptyset, X, \{1\}\}$ and $Y = \{a, b, c\}$ endowed with the topology $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. Let

$f : X \to Y$ he defined as follows: $f(0) = a$, $f(1) = b$. Then f is not weakly continuous at $x = 0$, but it is $\theta\delta$-continuous since the only θ-open set in Y is Y itself and $f^{-1}(Y) = X$ is δ-open.

Observe also that this function is θ-continuous but not weakly continuous at the point b. □

The following two theorems follow from definitions:

3.7. THEOREM. *Any $\theta\delta$-continuous function is faintly continuous.*

3.8. THEOREM. *If $f : (X, \tau) \to (Y, \sigma)$ is faintly continuous and X is semiregular, then f is $\theta\delta$-continuous.*

3.9. DEFINITION [CL]. *A function $f : X \to Y$ is said to be γ-continuous if for each $x \in X$ and each quasi-regular open subset V (i.e. an open subset containing a non empty regular closed subset) containing $f(x)$, there exists $U \in \mathcal{U}_x$ such that $f(U) \subset V$.*

3.10. REMARK. In general, $\theta\delta$-continuous functions are not γ-continuous because we have that almost continuity, θ-continuity and weakly θ-continuity imply γ-continuity, while these functions are independent [CL]. □

3.11. THEOREM. *Let $f : X \to Y$ be a γ-continuous function. If X is a semiregular space, then f is $\theta\delta$-continuous.*

PROOF. Let A be a θ-open subset of Y. Then A is a quasi-regular open subset, hence $f^{-1}(A)$ is open in X [CL]. But X is semiregular therefore $f^{-1}(A)$ is also δ-open. □

3.12. THEOREM. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $\theta\delta$-continuous function. If Y is an almost regular space, then f is δ-continuous (and so almost, θ-, weakly, faintly continuous).*

PROOF. Let $V \subset Y$ be a δ-open subset. Since Y is almost regular we have that $\sigma^* \equiv \sigma_\theta$ and so V is also θ-open. It follows that $f^{-1}(V)$ is δ-open. □

3.13. DEFINITION. *A function $f : X \to Y$ is called super continuous* [MB] (*resp. strongly θ-continuous* [N]) *if, for each $x \in X$ and each $V \in \mathcal{U}_{f(x)}$, there exists $U \in \mathcal{U}_x$ such that $f(\overset{\circ}{\overline{U}}) \subset V$ (resp. $f(\overline{U}) \subset V$). f is called almost γ-continuous* [CN] *if for each $x \in X$ and each $V \in \mathcal{U}(\overline{\mathcal{U}}_{f(x)})$ there exists $U \in \mathcal{U}_x$ such that $f(U) \subset V$.*

3.14. THEOREM. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $\theta\delta$-continuous function. If Y is a regular space, then f is super continuous (and so δ-, almost, θ-, weakly, faintly continuous, continuous, u-, γ-, almost γ-continuous).*

PROOF. For each $x \in X$ let $V \in \mathcal{U}_{f(x)}$ be an open neighbourhood of $f(x)$. Since Y is regular we have that V is θ-open, hence $f^{-1}(V)$ is δ-open, i.e. there exists $U \in \mathcal{U}_x$ such that $\overset{\circ}{\overline{U}} \subset f^{-1}(V)$. Then $f(\overset{\circ}{\overline{U}}) \subset V$, i.e. f is super continuous. □

3.15. Corollary. *Let $f : X \to Y$ be a function. If Y is a regular space, the following are equivalent:*

i) *f is super continuous;* ii) *f is δ-continuous;*
iii) *f is almost continuous;* iv) *f is θ-continuous;*
v) *f is weakly θ-continuous;* vi) *f is $\theta\delta$-continuous;*
vii) *f is continuous;* viii) *f is u-continuous.*

Proof. i) $\Longrightarrow$ii) $\Longrightarrow$iii) $\Longrightarrow$iv) $\Longrightarrow$v) $\Longrightarrow$vi) follow from definitions, vi)$\Longrightarrow$i) from Theorem 3.14, i) $\Longrightarrow$vii) $\Longrightarrow$viii) $\Longrightarrow$vi) from definitions. □

3.16. Theorem. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $\theta\delta$-continuous function. If X is an almost regular space, then f is u-continuous.*

Proof. For each θ-open set $V \subset Y$, $f^{-1}(V)$ is δ-open in X. But $\tau^* \equiv \tau_\theta$ (on X), so that $f^{-1}(V)$ is θ-open. □

3.17. Corollary. *Let $f : X \to Y$ be a function. If X is a regular space, then the following are equivalent:*

i) *f is u-continuous;*
ii) *f is $\theta\delta$-continuous;*
iii) *f is faintly continuous.*

Proof. i) $\Longrightarrow$ii) $\Longrightarrow$iii) follow from definitions, iii) $\Longrightarrow$ii) from Theorem 3.8, ii) $\Longrightarrow$i) from Theorem 3.12. □

3.18. Definition. *A topological space (X, τ) is called θ-regular if for each $x \in X$ we have $\mathcal{U}(\overline{\mathcal{U}}_x) = \theta\mathcal{U}_x$, i.e. if for each $A \in \mathcal{U}(\overline{\mathcal{U}}_x)$ there exists a θ-open subset B such that $x \in B \subseteq A$.*

3.19. Remark. It is obvious that any regular space is θ-regular since in this case $\mathcal{U}_x \equiv \theta\mathcal{U}_x$ and $\mathcal{U}_x \equiv \overline{\mathcal{U}}_x \equiv \mathcal{U}(\overline{\mathcal{U}}_x)$.

3.20. Proposition. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $\theta\delta$-continuous function. If Y is a θ-regular space, then f is almost γ-continuous.*

Proof. Let $x \in X$ and $V \in \mathcal{U}(\overline{\mathcal{U}}_{f(x)})$. By hypothesis there exists a θ-open subset $B \subset Y$ such that $f(x) \in B \subseteq V$. Since f is $\theta\delta$-continuous, $f^{-1}(B)$ is δ-open. Hence there exists $U \in \mathcal{U}_x$ such that $x \in \overset{\circ}{\overline{U}} \subseteq f^{-1}(B)$. Then $f(\overset{\circ}{\overline{U}}) \subseteq ff^{-1}(B) \subseteq B \subseteq V$. □

3.21. Corollary. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If Y is a regular space the following are equivalent:*

i) f is continuous; ii) f is almost continuous;
iii) f is θ-continuous; iv) f is weakly θ-continuous;
v) f is $\theta\delta$-continuous; vi) f is almost γ-continuous;
vii) f is faintly continuous.

Proof. i) $\Longrightarrow$ii) $\Longrightarrow$iii) $\Longrightarrow$iv) $\Longrightarrow$v) follow from definitions, v) $\Longrightarrow$vi) from Remark 3.19 and Proposition 3.20, vi) $\Longrightarrow$vii) from definitions and vii) $\Longrightarrow$i) from the fact that Y is regular and so $\sigma \equiv \sigma_\theta$. □

3.22. Corollary. *Let $f : X \to Y$ be a function. If Y is a regular space, then the conditions of Corollary 3.15 are equivalent to the following:*

i) *f is almost γ-continuous;* ii) *f is faintly continuous.* □

3.23. COROLLARY. *Let $f : X \to Y$ be a function. If Y is an almost regular space, then the following are equivalent:*

i) *f is faintly continuous;* ii) *f is almost continuous;*
iii) *f is θ-continuous;* iv) *f is weakly θ-continuous;*
v) *f is $\theta\delta$-continuous.*

PROOF. For i) $\Longrightarrow$ii) see [LH]; ii) $\Longrightarrow$iii) $\Longrightarrow$iv) $\Longrightarrow$v) follow from definitions and v) $\Longrightarrow$i) from Theorem 3.12. □

We recall that any continuous function is u-continuous [LH] and that any u-continuous function is $\theta\delta$-continuous (since any θ-open set is δ-open and open set).

4. Algebra of $\theta\delta$-continuous functions

4.1. THEOREM. *Let $f : X \to Y$ be a $\theta\delta$-continuous function. If $A \subset X$ is either open or dense in X then the restriction $f_A : A \to Y$ is $\theta\delta$-continuous.*

PROOF. Let V be a θ-open subset of Y. Then $f^{-1}(V)$ is δ-open in X. We prove that $f^{-1}(V) \cap A$ is δ-open in A. For each $x \in f^{-1}(V) \cap A$ (since $x \in f^{-1}(V)$) there exists a neighbourhood $U \in \mathcal{U}_x$ such that $\overset{\circ}{\overline{U}} \subset f^{-1}(V)$. But, since A is open or dense, we have that $A \cap \overset{\circ}{\overline{U}}$ is a regular open subset of A ([N$_1$], [C]) and, since $\overset{\circ}{\overline{U}} \cap A \subset f^{-1}(V) \cap A$, this completes the proof. □

4.2. THEOREM. *Let $f : X \to Y$ and $g : Y \to Z$ be two $\theta\delta$-continuous functions. If Y is an almost regular space, then $g \circ f$ is $\theta\delta$-continuous.*

PROOF. Let A be a θ-open subset of Z. Then $g^{-1}(A)$ is δ-open in Y and therefore θ-open. So $f^{-1}\big(g^{-1}(A)\big)$ is a δ-open set in X. □

4.3. PROPOSITION. *If $f : X \to Y$ is $\theta\delta$-continuous, $g : Y \to Z$ is faintly continuous and Y is a regular space, then $g \circ f$ is $\theta\delta$-continuous.* □

4.4. THEOREM. *Let $f : X \to Y$ be a $\theta\delta$-continuous function and $g : Y \to Z$ a u-continuous function. Then $g \circ f$ is $\theta\delta$-continuous.*

PROOF. Let A be a θ-open subset of Z. Then $g^{-1}(A)$ is θ-open in Y so that $f^{-1}\big(g^{-1}(A)\big)$ is δ-open in X. □

4.5. COROLLARY. *Let $f : X \to Y$ be a $\theta\delta$-continuous function and $g : Y \to Z$ a continuous function. Then $g \circ f$ is $\theta\delta$-continuous.* □

4.6. THEOREM. *Let $f : X \to Y$ be a $\theta\delta$-continuous function (resp. almost strongly θ-continuous) and $g : Y \to Z$ an almost strongly θ-continuous function (res. $\theta\delta$-continuous). Then $g \circ f$ is δ-continuous (resp. u-continuous).* □

4.7. THEOREM. *Let $f_i : X_i \to Y_i$ be functions for $i = 1, 2, \dots, n$ and let $f : \prod_i X_i \to \prod_i Y_i$ be the function defined by $f\big(\{x_i\}\big) = \{f_i(x_i)\}$ for each point $\{x_i\} \in \prod_i X_i$. Then f is $\theta\delta$-continuous if and only if f_i is $\theta\delta$-continuous for each i.*

PROOF. Let W_i be a θ-open subset of Y_i for each $i = 1, 2, \dots, n$. Then $\prod_i W_i$ is θ-open in $\prod_i Y_i$ [LH]. It follows that $f^{-1}(\prod_i W_i) = \prod_i f_i(W_i)$ is δ-open. So, for each $\{x_i\} \in f^{-1}(\prod_i W_i)$, there exists a neighbourhood $U = \prod_i U_i$ such that $\{x_i\} \in \overset{\circ}{\overline{U}} \subset f^{-1}(\prod_i W_i)$. But $\overset{\circ}{\overline{U}} = \mathrm{int}(\mathrm{cl}(\prod_i U_i)) = \prod_i \overset{\circ}{\overline{U}}_i \subset f^{-1}(\prod_i W_i) = \prod_i f_i{}^{-1}(W_i)$, i.e. for each $i = 1, 2, \dots, n$, $\overset{\circ}{\overline{U}}_i \subset f_i{}^{-1}(W_i)$, so $f_i{}^{-1}(W_i)$ is δ-open for each $i = 1, 2, \dots, n$. Conversely, if W is a θ-open subset of $\prod_i Y_i$, then by [LH] there exist θ-open sets $W_1, W_2, \dots, W_n$ such that $W_i \subset Y_i$ for each $i = 1, 2, \dots, n$ and $W = \prod_i W_i$. Consider $f^{-1}(W) = f^{-1}(\prod_i W_i) = \prod_i f_i{}^{-1}(W_i)$. By hypothesis $f_i{}^{-1}(W_i)$ is δ-open in X_i for each $i = 1, 2, \dots, n$ and therefore the set $\prod_i f_i{}^{-1}(W_i)$ is δ-open. The proof is complete. □

4.8. THEOREM. *A function $f : X \to \prod_i X_i$ is $\theta\delta$-continuous if and only if $p_i \circ f : X \to X_i$, where p_i is the projection from $\prod_i X_i$ in X_i, is $\theta\delta$-continuous for each i.*

PROOF. By Corollary 4.5 it is sufficient to prove that if $p_i \circ f$ is $\theta\delta$-continuous for each i, then f is $\theta\delta$-continuous. Let A be a θ-open subset of $\prod_i X_i$. Denote $g = p_i \circ f$. Then $f^{-1}(A) = g^{-1}p_i(A)$. The subset $B = p_i(A)$ is θ-open [LH], g is $\theta\delta$-continuous and hence $g^{-1}(B) = f^{-1}(A)$ is δ-open. This completes the proof. □

4.9. THEOREM. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function defined by $g(x) = (x, f(x))$ for each $x \in X$. Then f is $\theta\delta$-continuous if and only if g is $\theta\delta$-continuous.*

PROOF. Let V be a θ-open subset of $X \times Y$. Then $V = V_1 \times V_2$, where V_1 and V_2 are θ-open subsets of X and Y respectively. We have that $g^{-1}(V) = g^{-1}(V_1 \times V_2) = \{ x \in V_1 : f(x) \in V_2 \} = V_1 \cap f^{-1}(V_2)$. We have to prove that $V_1 \cap f^{-1}(V_2)$ is δ-open. Since V_1 is θ-open, for each $x \in V_1 \cap f^{-1}(V_2)$, there exists $H \in \mathcal{U}_x$ such that $\overline{H} \subset V_1$, hence $\overset{\circ}{\overline{H}} \subset V_1$; moreover $f^{-1}(V_2)$ is δ-open so that there exists a neighbourhood $G \in \mathcal{U}_x$ such that $\overset{\circ}{\overline{G}} \subset f^{-1}(V_2)$. Consider $H \cap G$ as a neighbourhood of x; then $\mathrm{int}(\overline{H \cap G}) = \overset{\circ}{\overline{H}} \cap \overset{\circ}{\overline{G}} \subset V_1 \cap f^{-1}(V_2)$, i.e. $V_1 \cap f^{-1}(V_2)$ is δ-open. Conversely, let V be a θ-open subset of Y. $X \times V$ is θ-open in $X \times Y$ and thus $g^{-1}(X \times V) = \{ x \in X : f(x) \in V \} = f^{-1}(V)$ is δ-open in X, which completes the proof. □

REFERENCES

[AD] S. P. ARYA, M. DEB, *On θ-continuous mappings*, The Math. Student, 42, (1974), 81–89.

[BC] A. BELLA, F. CAMMAROTO, *On the cardinality of Urysohn spaces*, Canad. Math. Bull., 31, (1988), 153–158.

[C] F. CAMMAROTO, *On δ-continuous and δ-open functions*, Kyungpook Math. J., 28, (1988).

[C$_1$] F. CAMMAROTO, *Filtri particolarmente chiuisi e particolarmente aperti*, Le Matematiche Univ. di Catania, 32 fasc. II (1977), 343–358.

[CL] F. Cammaroto, G. Lo Faro, *Sulle funzioni γ-continue*, Le Matematiche (Catania), 35 (1980), 1–17.

[CN] F. Cammaroto, T. Noiri, *Almost γ-continuous functions* - to appear on Int. J. Math. Soc (USA), 1990 (to appear).

[CN$_1$] F. Cammaroto, T. Noiri, *On weakly θ-continuous functions*, Matem. vesnik, 38 (1986), 33-44.

[G] E. Guili, D. Dikranjan, *$S(n)-\theta$-closed spaces*, Topology Appl., 28 (1988) n.1, 59–74.

[L] N. Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly, 68 (1961), 44-46.

[LH] P. E. Long, L. L. Herrington, *The τ_θ-topology and faintly continuous functions*, Kyungpook Math. J., Vol. 22, (1982), 7–14.

[MB] B. M. Munshi, D. S. Bassan, *Super continuous mappings*, Indian J. Pure Appl. Math., 13 (1982), 229–236.

[MR] M. Mršević, I. L. Reilly, *On weakly θ-continuous functions*, to appear.

[N] T. Noiri, *On δ-continuous functions*, J. Korean Math. Soc. 16 (1980), 161–166.

[N$_1$] T. Noiri, *On almost-open mappings*, Mem. Miyakonojo Tech. Coll., 7 (1972), 167–171.

[NS] T. Noiri, Sin Min Kang, *On almost strongly θ-continuous functions*, Indian J. Pure Appl. Math., 15 (1984), 1–8.

[SA] M. K. Singal, S. P. Arya, *On almost regular spaces*, Glasnik Mat. 4(24) (1969), 89–99.

[SS] M. K. Singal, A. R. Singal, *Almost-continuous mappings*, Yokohama Math. J., 16 (1968), 63–73.

[V] N. V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl., (2) 78 (1968), 103–118.

Grazia Santoro
Dipartimento di Matematica-Universita´ di Messina
Salita Sperone n. 31
98 166 Sant´Agata - MESSINA (Italia).