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SCC AND HCC EXTENSIONS AND CONTINUOUS IMAGES OF HCC SPACES

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ABSTRACT. *In this paper we further investigate the results given in* [9] *and* [10]. *A space X is HCC (hypercountably compact) if every σ-compact set in X has the compact closure in X (see* [10]). *A space X is SCC(strongly countably compact) if every countable subset in X has the compact closure in X (see* [6]). *A pair (Y, c) is called a SCC(HCC) extension of the space X if Y is a SCC(HCC) space and $c : X \longrightarrow Y$ is a homeomorphic embeding of X in Y such that $cl_Y(c(X)) = Y$. In section 2 we consider SCC and HCC extensions of locally compact spaces. In section 3 we also consider continuous images of HCC spaces.*

1. Introduction

The closure of A, a subset of a space X, is denoted by $cl_X(A)$. In this paper we assume that all spaces are Hausdorff (T_2-spaces). For notation and definitions not given here see [5,6,7].

Let X be a topological T_2-space. Then:
(1) $\mathcal{K}(X)$ denotes the set of all non-empty compact subspaces of X.
(2) $\mathcal{F}(X) = \{F \subset X : F \text{ is finite}\} \subset \mathcal{K}(X)$.

DEFINITION 1.1. A space X is strongly countably compact (SCC) if the closure of every countable subset in X is compact in X (see [6]).

DEFINITION 1.2. A space X is hypercountably compact (HCC) if the closure of the union of every countable family of compact sets in X is a compact set in X (see [10]).

DEFINITION 1.3. Let X be a topological T_2 - space.
(1) A point $p \in X$ is a said to be a P-point provided that intersection of countably many neighbourhoods of p is a neighbourhood of p (see[7]).
(2) A point $p \in X$ is a weak P-point if $p \notin cl_X(F)$ for any countable $F \subset X - \{p\}$ (see [7]).

It is easy to see that every P-point is a weak P-point. The converse is not necessarly true (see [10] example 2.1, [7]).

PROPOSITION 1.4. *Let X be a locally compact T_2-space and let ωX denote the one-point compactification of X. Then:*
(1) The space X is a SCC space if and only if the point $x_0 = \omega X - X$ is a weak P-point.
(2) The space X is a HCC space if and only if the point $x_0 = \omega X - X$ is a P-point.

PROOF. (1): Suppose that X is an SCC space and let $A \subset X = \omega X - \{x_0\}$ be a countable set. Then $cl_{\omega X}(A) = cl_X(A)$ is a compact set in X and $x_0 \notin cl_{\omega X}(A)$. Hence point x_0 is a weak P-point in ωX.

Conversely, suppose that x_0 is a weak P-point in ωX. Let $A \subset X = \omega X - \{x_0\}$ be a countable set. Then $x_0 \notin cl_{\omega X}(A)$; hence $cl_{\omega X}(A) = cl_X(A)$. Since $cl_{\omega X}(A)$ is

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compact subspace in ωX, $cl_X(A)$ is a compact subspace in $X = \omega X - \{x_0\}$. Hence, subspace $X = \omega X - \{x_0\}$ is an SCC space.

(2): Suppose that X is HCC space and let $\{U_n(x_0) : n \in N\}$ be any countable family of open neighbourhoods of the point x_0 in ωX. Then $\cap\{U_n(x_0) : n \in N\} = \cup\{\omega X - U_n(x_0) : n \in N\}$ and $\omega X - U_n(x_0) \subset X$ is a compact subspace for each $n \in N$. Since X is an HCC space there exists a compact set $K \subset X$ such that $\omega X - U_n(x_0) \subset K$ for each $n \in N$. The set $U = \omega X - K$ is an open neighbourhood of x_0 in ωX. Therefore, $\omega X - K \subset \cap\{U_n(x_0) : n \in N\}$. Hence, x_0 is a P-point in X.

Conversely, suppose that x_0 is a P-point in ωX and let $\{K_n : n \in N\}$ be a countable family of compact sets in X. Then $\cup\{K_n : n \in N\} = \cap\{\omega X - K_n : n \in N\}$ and $\omega X - K_n$ is an open neighbourhood of x_0 in ωX for aech $n \in N$. Since x_0 is a P-point in ωX, there exists an open neighbourhood U of x_0 such that $U \subset \cap\{\omega X - K_n : n \in N\}$. Therefore, $K_n \subset \omega X - U$ for each $n \in N$ and $\omega X - U$ is a compact subset of X. Hence, by Definition 1.2, X is an HCC space. This completes the proof.

Corollary 1.5. *Let be a Tychonoff space for which $\beta X - X = \{x_0\}$, (i.e. $\omega X = \beta X$). The space X is HCC(SCC) if and only if the point x_0 is a P-point (weak P-point).*

Definition 1.6. A pair (Y, c), is called a SCC(HCC) extension of a space X, if Y is a SCC(HCC) space and $c : X \longrightarrow Y$ is a homeomorphic embedding of X in Y such that $cl_Y(c(X)) = Y$.

If a space X is embeddable in a SCC(HCC) space Y, i.e., if there exists a homeomorphism $f : X \longrightarrow M$ onto a subspace $M = f(X)$ of Y, then obviously the pair $(cl_Y((fX)), i \circ f)$, where i denotes the embedding of M in $cl_Y(M)$, is a SCC(HCC) extension of the space X. Hence every space which is embeddable in a SCC(HCC) space has a SCC(HCC) extension.

2. SCC and HCC extensions of locally compact spaces

Proposition 2.1. *Let X be a locally compact and pseudocompact space which is not a SCC space. Then X can be embedded as an open subspace into an SCC space σX.*

Proof. Let τ be the topology of X and let $\mathcal{X} = \{A : A \subset X, card(A) = \aleph_0$ and $cl_X(A) \notin \mathcal{K}(X)\}$. Since X is not a SCC space, $\mathcal{X} \neq \emptyset$. Let ρ be the equivalence relation on $\mathcal{X}$ defined by $A\rho B \Leftrightarrow cl_X(A) = cl_X(B)$ $(A \in \mathcal{X}, B \in \mathcal{X})$. $\mathcal{X} = \cup\{\mathcal{C}_i : \mathcal{C}_i$ is an equivalence class of ρ, $i \in D\}$, $\mathcal{X}/\rho = \{\mathcal{C}_i : i \in D\}$, where D is an index set for $\mathcal{X}/\rho$. One can note that for every $\mathcal{C}_i$, $i \in D$, there exists a noncompact closed subspace $X_i \subset X$, $i \in D$ with the property that for every $A \in \mathcal{C}_i$, $cl_X(A) = X_i$, $i \in D$. Let $X^* = \{X_i : i \in D\}$ and $X' = X \cup X^*$. Obviosly $X \cap X^* = \emptyset$. We now introduce a topology τ' on X' as follows: $\tau' = \tau \cup \{\{X_i\} \cup (X_i - K) : K \in \mathcal{K}(X_i), i \in D\}$ where τ is the topology on X and $\{\{X_i\} \cup (X_i - K) : K \in \mathcal{K}(X), i \in D\}$ are basic neighborhoods of point $X_i \in X^*$ intersecting X. Let $\sigma X = X' \cup \{p\}$ where $p \notin X'$ and σ is an topology on σX defined as follows:

$$\sigma = \tau' \cup \{\{p\} \cup (X^* - F) : F \in \mathcal{K}(X^*)\}$$

The space $(\sigma X, \sigma)$ has the following properties:

The subspace X is open in σX and $cl'_X(X) = X'$.
Since X is locally compact this is obvious from the way we constructed topology σ.

The subspace $\sigma X - X$ is closed and compact in σX.
This is clear, because subspace $\sigma X - X$ is the one-point compactification of X^*.

The point $p \in \sigma X$ is an isolated point of X.

This is clear, because $(\{p\} \cup X^*) \cap X = \emptyset$.

The space $(\sigma X, \sigma)$ is SCC but it is not compact.

Let $A = \{a_n \in X : n \in N\}$ be any countable subset of $X \subset \sigma X$ and let $A \in \mathcal{X}$. Then there exists an $X_{i_0} \in X^*$ such that $cl_X(A) = X_{i_0}$. The set $X_{i_0} \cup \{X_{i_0}\}$ is compact in σX and $cl_{\sigma X}(A) = X_{i_0} \cup \{X_{i_0}\}$. Suppose that $A \subset \sigma X - X$. Then $cl_{\sigma X - X}(A) = cl_{\sigma X}(A) \in \mathcal{K}(\sigma X)(\sigma X - X$ is closed and compact in $\sigma X)$. Hence σX is a SCC space. Since X is an open subspace in σX and it is not compact there exists an open infinite cover $\mathcal{V} \subset \tau$ without a finite subcover. Then $\mathcal{V} \cup (\{p\} \cup X^*)$ is an open cover of σX without a finite subcover. Hence $(\sigma X, \sigma)$ is not compact.

Let $x_1 \in X$ and $x_2 \in X$, $x_1 \neq x_2$, $X \subset \sigma X = X \cup X^* \cup \{p\}$. Since X is Hausdorff and X is open in σX, any two distinct points in X have disjoint neighborhoods in σX.

Let $x_1 \in X$ and $x_2 \in X^*$, i.e., $x_2 = X_{i_0} \in X^*$. Since X is locally compact, there exists an open neighborhood G of x_1 in X such that $cl_X(G) \in \mathcal{K}(X)$. The set $cl(G) \cap X_{i_0} \in \mathcal{K}(X_{i_0}) \subset \mathcal{K}(X)$. Let $U_1 = G$ and $U_2 = \{X_{i_0}\} \cup (X_{i_0} - (cl_X(G) \cap X_{i_0}))$. Then U_1 and U_2 are disjoint neighborhoods of x_1 and x_2 in σX.

Let $x_1 \in X$ and $x_2 = p$; then there exists an open neighborhood G of x_1 in X such that $cl_X(G) \in \mathcal{K}(X)$ (X is locally compact). The sets G and $\{p\} \cup X^*$ are disjoint neighborhoods of x_1 and p in σX.

Let $x_1 = X_{i_1} \in X^*$ and $x_2 = p$; then $\{X_{i_1}\} \cup X_{i_1}$ and $\{p\} \cup (X^* - \{X_{i_1}\})$ are disjoint neighborhoods of X_{i_1} and p in σX.

Let $x_1 = X_{i_1}$ and $x_2 = X_{i_2}$ be distinct points in X^*; then $V_1 = \{X_{i_1}\} \cup X_{i_1}$ and $V_2 = \{X_{i_2}\} \cup X_{i_2}$ are neighborhoods of x_1 and x_2 in σX such that $x_1 \notin V_2$ and $x_2 \notin V_1$.

REMARKS. (a) Let $\mathcal{S}(X) = \{A : A \subset X,\ A$ is a closed separable set$\}$. If $(A \cap B) \in \mathcal{K}(X)$, for any two A and B in $\mathcal{S}(X)$, then σX is a Hausdorff space.

(b) Let $\text{card}(X^*) \prec \aleph_0$. Then $\sigma X = X'$ and $\sigma = \tau'$, i.e., $(\sigma X, \sigma) = (X', \tau,)$. It is easy to see that $cl_{\sigma X}(X) = \sigma X$.

(c) Let familly $\mathcal{X} = \{A : A \subset X, card(A) = \aleph_0$ and $cl_X(A) \notin \mathcal{K}(X)\}$ be totaly-ordered by inclusion. If there exists a maximal element $M \in \mathcal{X}$, then $\sigma X = \tau \cup \{\{cl_X(M)\} \cup (cl_X(M) - K) : K \in \mathcal{K}(cl_X(M))\}$. The space σX is Hausdorff and $cl_{\sigma X}(X) = \sigma X$.

For example, the deleted Tychonoff plenk $X = [0, \omega_1] \times [0, \omega_0] - \{(\omega_1, \omega_0)\}$ is a pseudocompact and locally compact space which is not SCC. The subset $A \subset X$ where $A = \{(\omega_1, n) : 0 \leq n \prec \aleph_0\}$ is a maximal element for $\mathcal{X}$ and $cl_X(A) = A$. Hence $\sigma X = X \cup \{A\}$ and $\sigma = \tau \cup \{\{A\} \cup (A - K) : K \in \mathcal{K}(A)\}$.

(d) Since X is locally compact and Hausdorff it is Tychonoff. Therefore, there exist compactifications ωX ($\omega X = X \cup \{x_0\}; x_0 \notin X$) and βX. Let $y \in \beta X - X$. If y is a weak P-point, then the subspace $\beta X - \{y\}$ is a SCC space (see [10], 3.3), i.e., $\beta X - \{y\}$.

(e) By 1.4, the one-point compactification ωX is an SCC extension of X.

PROPOSITION 2.2. *Let X be a locally compact and SCC space which is not a HCC space. Then X can be imbedded as an open subspace of HCC space φX.*

PROOF. Let τ be the topology on X and let $\mathcal{X} = \{A : A \subset X,\ A$ is $\sigma-compact$ and $cl_A(X) \notin \mathcal{K}(X)\}$. Since X is not a HCC space, $\mathcal{X} \neq \emptyset$. Let ρ be the equivalence relatiom on $\mathcal{X}$ defined by $A \rho B \Leftrightarrow cl_X(A) = cl_X(B)$ $(A \in \mathcal{X}, B \in \mathcal{X})$. $\mathcal{X} = \cup\{\mathcal{C}_i : \mathcal{C}_i$ is an equivalence class of $\rho, i \in D\}$, $\mathcal{X}/\rho = \{\mathcal{X}_i : i \in D\}$ where D is an index set for $\mathcal{X}/\rho$. One can note that for every $\mathcal{C}_i, i \in D$ there exists a noncompact closed subspace $X_i \subset X, i \in D$ with the property that for evert $A \in \mathcal{C}_i$ $cl_X(A) = X_i$, $i \in D$. Let $X^* = \{X_i : i \in D\}$ and $X' = X \cup X^*$. Obviosly, $X \cap X^* = \emptyset$. We introduce a topology τ' on X' as follows:

$$\tau' = \tau \cup \{\{X_i\} \cup (X_i - K) : K \in \mathcal{K}(X_i); i \in D\}$$

where τ is a topology of X and $\{\{X_i\} \cup (X_i - K) : K \in \mathcal{K}(X_i); i \in D\}$ are basic neighborhoods of points $X_i \in X^*$ intersecting X. Let $\varphi X = X' \cup \{p\}$ where $p \notin X'$ and φ a topology on φX defined as follows:

$$\varphi = \tau' \cup \{\{p\} \cup (X^* - F) : F \in |CalK(X^*)\}$$

The space $(\varphi X, \varphi)$ has the following properties:
The subspace X is open in φX and $cl_{X'}(X) = X'$.
The subspace $\varphi X - X$ is closed and compact in φX.
The point $p \in \varphi X$ is an isolated point for X.
The space $(\varphi X, \varphi)$ is HCC but it is not compact.

Let A be any σ-compact subset of $X \subset \varphi X$ ($\varphi X = X \cup X^* \cup \{p\}$) and let $A \in \mathcal{X}$. Then there exists an $X_{i_0} \in X^*$ such that $cl_X(A) = X_{i_0}$. The set $X_{i_0} \cup \{X_{i_0}\}$ is compact in φX and $cl_{\varphi X} = X_{i_0} \cup \{X_{i_0}\}$. Suppose that $A \subset \varphi X - X$. Then $cl_{\varphi X - X}(A) = cl_{\varphi X}(A) \in \mathcal{K}(\varphi X)$ ($\varphi X - X$ is closed and compact in φX). Hence φX is a HCC space. Since the subspace $X \subset \varphi X$ is open in φX and it is not compact, there exists an open infinite cover $\mathcal{V}$ without a finite subcover. Then $\mathcal{V} \cup (\{p\} \cup X^*)$ is an open cover of φX without a finite subcover. Hence $(\varphi X, \varphi)$ is not compact.

Let x_1 and x_2 be any two distinct points in φX. If $x_1 \in X$, $x_2 \in X$; $x_1 \in X$, $x_2 \in X^*$; $x_1 \in X$, $x_2 = p$; $x_1 \in X^*$, $x_2 = p$; then there exist disjoint neighborhoods of x_1 and x_2 in φX (see proof of 2.1). Let $x_1 = X_{i_1}$ and $x_2 = X_{i_2}$ be distinct points in X^*. Then $U_1 = X_{i_1} \cup \{X_{i_1}\}$ and $U_2 = X_{i_2} \cup \{X_{i_2}\}$ are neighborhoods of x_1 and x_2 such that $x_1 \notin U_2$ and $x_2 \notin U_1$.

REMARKS (a) Let $\mathcal{R}(X) = \{A : A \subset X;\ A$ is a closure of a σ-compact set$\}$. If $(A \cap B) \in \mathcal{K}(X)$, for any two A, B in $\mathcal{R}(X)$, then φX is a Hausdorff space.

(b) Let card$(X^*) \prec \aleph_0$. Then $\varphi X = X'$ and $\varphi = \tau'$, i.e., $(\varphi X, \varphi) = (X', \tau')$. It is easy to see that $cl_{\varphi X}(X) = \varphi X$.

(c) The familly $\mathcal{X} = \{A : A \subset X,\ A$ is a σ-compact and $cl_X(A) \notin \mathcal{K}(X)\}$ is totaly ordered by inclusion. If there exist a maximal element $M \in \mathcal{X}$, then $\varphi X = X \cup \{cl_X(M)\}$ and $\varphi = \tau \cup \{\{cl_X(M) \cup (cl_X(M) - K) : K \in \mathcal{K}(cl_X(M))\}$. The space φX is Hausdorff and $cl_{\varphi X}(X) = \varphi X$.

(d) Since X is locally compact and Hausdorff it is Tychonoff. Therefore, there exist compactifications ωX ($\omega X = X \cup \{p\}, p \notin X$) and βX. Let $p \in \beta X - X$. If p is a P-point, then the subspace $\beta X - \{p\}$ is an HCC space(see [10], 3.3), i.e., $\beta X - \{p\} = \varphi X$ and φ is topology of $\beta X - \{p\}$ induced by βX.

(e) By proposition 1.4, the one-point compactification ωX is HCC extension of X.

(f) Let X be a locally compact and SCC space which is not a HCC space. Then by 2.2, X can be imbedded as an open subspace of an HCC space φX. The space $X \cup \{y\}$, where y is an isolated point in X, is an SCC space. Then the function $f : \varphi X \longrightarrow X \cup \{y\}$ defined as

$$f(x) = \begin{cases} x, & x \in X \\ y, & x \in \varphi X - X \end{cases}$$

is a continuous function from φX to the SCC space $X \cup \{y\}$. If $y \in X$, then f is continuous function from φX onto X.

3. Continuous images of HCC spaces

It can be shown that every continuous image of an HCC space is an SCC space. The HCC property is not a continuous invariant(see [10] example 2.4).

DEFINITION 3.1. A space X is a C_H-space if there exists an HCC space Y and continuous mapping $f : Y \longrightarrow X$ from Y onto X.

It is clear that every C_H-space is an SCC space. By remark (f) in 2.2, the converse is not necessarily true.

PROPOSITION 3.2. *Every closed subspace of a C_H-space is a C_H-space.*

PROOF. Let Y be a closed subspace of C_H-space X. By 3.1, there exists an HCC space W and a continuous mapping $f : W \longrightarrow X$ from W onto X. The set $f^{-1}(Y)$ is closed in W. By 2.5 in [10], $f^{-1}(Y)$ is an HCC space. Hence Y is a C_H-space. This completes the proof.

PROPOSITION 3.3. *A continuous image of a C_H-space is a C_H-space.*

PROOF. Let X be a C_H-space and $f : X \longrightarrow Y$ a continuous mapping. Let $g : W \longrightarrow X$ be a continuous mapping from an HCC space W onto X. Then, since g is continuous, $g \circ f$ is a continuous mapping from HCC space W onto Y. Hence, Y is a C_H-space. This completes the proof.

COROLLARY 3.4. *Every quotient space of a C_H-space is a C_H-space.*

PROPOSITION 3.5. *The disjoint topological sum of a finite family of C_H-space is a C_H-space.*

PROOF. Let X be the disjoint sum of the family $\{X_i : i \in A \; card(A) \prec \aleph_0\}$, of C_H-spaces. Then for each $i \in A$ we have an HCC space Y_i and a cotinuous surjection $f_i : Y_i \longrightarrow X_i$. Let Y be the disjoint sum of the family $\{Y_i : i \in A \; card(A) \prec \aleph_0\}$. Then, by proposition 2.6, in [10] Y is an HCC space. The mapping $f : Y \longrightarrow X$ defined as $f(Y) = f_i(Y)$; $Y = Y_i$, $i \in A$, is a continuous surjection. Hence, X is a C_H-space. This completes the proof.

PROPOSITION 3.6. *Let $\{X_a : a \in A\}$ be a family of non-empty spaces. Then the product space $X = \prod\{X_a : a \in A\}$ is a C_H-space if and only if X_a is a C_H-space for each $a \in A$.*

PROOF. Let X be the product of spaces X_a , $a \in A$. If X is C_H-space, then every X_a , $a \in A$, is a C_H-space. This is a direct consequence of Proposition 3.3.

Conversely, suppose that every X_a , $a \in A$, is a C_H-space. Then for each $a \in A$ we have HCC space Y_a and a continuous mapping f_a from Y_a onto X_a. The product space $Y = \prod\{Y_a : a \in A\}$ is an HCC space (see 2.8, in [10]). The mapping $f : Y \longrightarrow X$ defined by formula $\forall y \in Y(y = \{y_a : a \in A\} \Rightarrow f(y) = \{f_a(y_a) : a \in A\} = x \in X)$ is a continuous mapping from Y onto X. Hence, X is a C_H-space. This completes the proof.

The following is an immediate consequence of Propositions 3.2 and 3.6.

COROLLARY 3.7. *The limit of an inverse system of C_H-spaces is a C_H-space.*

PROPOSITION 3.8. *Let X be a first countable C_H-space. Then X is a HCC space.*

PROOF. This is a direct consequence of Theorem 2.3 in [10] and remark 3.1.

PROPOSITION 3.9. *Let X be a separable C_H-space. Then X is a compact space.*

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