

# FEW OBSERVATIONS ON TOPOLOGICAL SPACES WITH SMALL DIAGONAL

A. V. ARHANGEL'SKII, A. BELLA

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**ABSTRACT.** *The aim of the paper is to present some results concerning the class of spaces having a small diagonal.*

The diagonal  $\Delta$  of a set  $X$ , also denoted by  $\Delta(X)$  when some confusion can occur, is the set  $\{(x, x) : x \in X\}$ . A topological space  $X$  is said to have  $\kappa$ -inaccessible diagonal, provided that for any set  $A \subset X^2 \setminus \Delta$  with  $|A| = \kappa$  there exists a neighbourhood  $U$  of  $\Delta$  for which  $|A \setminus U| = \kappa$ . This class of spaces was first studied in [5]. A space with  $\aleph_1$ -inaccessible diagonal is also called a space with small diagonal. Various results connected with this special case can be found in [3], [8] and [6].

A basic question is whether it is true that a compact space with small diagonal is metrizable. A positive answer was given in [8] under a complicated set theoretical assumption and another, using (CH) and the hypothesis that the spaces in consideration are of countable tightness, or satisfy some other strong restrictions on cardinal invariants, was given in [5]. Recently Juhász and Szentmiklóssy have proved that (CH) only is enough to guarantee that a compact space with small diagonal is metrizable.

In this paper, among other things, we show that a compact monolithic space with small diagonal is metrizable and that a cleavable space has small diagonal.

Our notation is standard and follows [4]. By  $\chi(\Delta, X^2)$  and  $\psi(\Delta, X^2)$  we denote respectively the character and the pseudo character of  $\Delta$  in  $X^2$ . Compact means compact Hausdorff and all spaces in the sequel assumed to be at least  $T_1$ .

A key point in the proof of the various results mentioned at the beginning is to show that a given compact space with small diagonal has actually a  $G_\delta$  diagonal. It is interesting, however, to find conditions under which this fact occurs in more general situations.

**THEOREM 1.** *If  $X$  is a regular space with small diagonal,  $X^2$  has the Lindelöf property and  $\psi(\Delta, X^2) \leq \aleph_1$  then  $X$  has a  $G_\delta$  diagonal.*

**PROOF.** Let  $\{U_\alpha : \alpha \in \omega_1\}$  be a family of open subsets of  $X^2$  such that  $\bigcap_{\alpha \in \omega_1} U_\alpha = \Delta$ . Because of the normality of  $X^2$ , we can assume that also  $\bigcap_{\alpha \in \omega_1} \overline{U_\alpha} = \Delta$ . If for some  $\beta \in \omega_1$  we have  $\bigcap_{\alpha \in \beta} U_\alpha = \Delta$  then we are done. Thus assume the contrary and

for any  $\beta \in \omega_1$  pick a point  $x_\beta \in \bigcap_{\alpha \in \beta} U_\alpha \setminus \Delta$ . The set so obtained has cardinality  $\aleph_1$  and converges to  $\Delta$ . This is a contradiction and the proof is complete.

**COROLLARY 1.** (CH) Any Lindeloff  $p$ -space with small diagonal and weight  $\aleph_1$  is metrizable.

**COROLLARY 2.** (CH) Any Lindeloff  $\Sigma$ -space with small diagonal and weight  $\leq \aleph_1$  has a countable network.

**COROLLARY 3.** If  $X$  is a space with small diagonal then no compact subspace of  $X$  has weight  $\aleph_1$ .

Recall that a space  $X$  is said to be monolithic provided that  $nw(\overline{A}) \leq |A|$  for any  $A \subset X$ .

**COROLLARY 4.** Every compact monolithic space with small diagonal is metrizable.

**PROOF:** If the space is not separable then we can select a left separated subset  $A$  of it such that  $|A| = \aleph_1$ . Since the space is compact and monolithic, it follows that the subspace  $\overline{A}$  has weight  $\aleph_1$  in contradiction with Corollary 3. This shows that the space must be separable and hence metrizable.

In connection with Corollary 1 we mention the following:

**QUESTION 1.** Is it true (assuming (CH) or in ZFC) that any paracompact  $p$ -space with small diagonal and weight  $\aleph_1$  is metrizable?

Recall that a space  $X$  is said to be cleavable (see [2]) provided that for any set  $A \subset X$  there exists a continuous map  $f : X \rightarrow \mathbb{R}^{\aleph_0}$  such that  $A = f^{-1}(f(A))$ . This notion is linked to  $C_p$ -theory: a Tychonoff space  $X$  is cleavable if and only if  $[C_p(X)]_{\aleph_0} = \mathbb{R}^X$ .

It is easy to see that any point of a cleavable space is a  $G_\delta$ , but it is not clear whether every such Tychonoff space has a  $G_\delta$  diagonal. A partial answer in this direction is in the next theorem:

**LEMMA 1.** If  $X$  is cleavable and  $A$  is a subset of  $X$  with  $|A| \leq 2^{\aleph_0}$  then there exists a continuous map  $f : X \rightarrow \mathbb{R}^{\aleph_0}$  which is one to one on  $A$ .

**THEOREM 2.** Any cleavable space has small diagonal.

**PROOF:** Let  $A$  be a subset of  $X^2 \setminus \Delta$  with  $|A| = \aleph_1$ . Let  $B = \cup\{(x, y) : (x, y) \in A\}$  and observe that  $|B| = \aleph_1$ . Select a continuous map  $f : X \rightarrow \mathbb{R}^{\aleph_0}$  which is one to one on  $B$ . The map  $g : X^2 \rightarrow \mathbb{R}^{\aleph_0}$  defined by  $(x, y) \rightarrow (f(x), f(y))$  is clearly one to one on  $A$  and  $g(A) \cap \Delta(\mathbb{R}^{\aleph_0}) = \emptyset$ . Because  $\mathbb{R}^{\aleph_0}$  has a  $G_\delta$  diagonal there is a neighbourhood  $U$  of  $\Delta(\mathbb{R}^{\aleph_0})$  such that  $|g(A) \setminus U| = \aleph_1$  and consequently also  $|A \setminus g^{-1}(U)| = \aleph_1$ . This shows that  $X$  has small diagonal.

Since the class of spaces with small diagonal is countably productive, the same argument as in the proof above actually shows that the following assertion is true:

**PROPOSITION 1.** Any space which is cleavable over the class of spaces with small diagonal has small diagonal.

Theorem 2 follows from proposition 2 since  $\mathfrak{R}^{\aleph_0}$  has a  $G_\delta$  diagonal and in fact the same proof shows that any cleavable space has  $\kappa$ -inaccessible diagonal, for any regular  $\kappa$  not exceeding  $2^{\aleph_0}$ .

Notice that a space with small diagonal is in general far from being cleavable. Indeed even a metrizable space can fail to be so (see[2]). In connection with the question posed above, it is perhaps worthwhile to observe that a cleavable paracompact p-space is metrizable (see[2]).

Let us mention a result of the same sort as Theorem 2 which provides a sufficient condition for a space to have small diagonal.

**PROPOSITION 2.** *If any subspace of  $X$  of density not exceeding  $\aleph_1$  has small diagonal then  $X$  has small diagonal.*

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Moscow State University, Mech.-Math. Fac., Russia

Universita di Messina, Dipartimento di Matematica