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SYMMORPHIC SPACE GROUPS OF SIMPLE AND MULTIPLE COLORED ANTISYMMETRY

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Abstract. *For all nontrivial cases of assigning to points of three-dimensional Euclidean space colored by p colors signs + or -, theoretical background and survey of complete derivation of junior symmorphic groups of colored simple and multiple antisymmetry of different patterns, is given.*

0. Introduction

Shubnikov teaching of antisymmetry [1] is used as the basis for various new generalizations of the classical theory of symmetry and their large application in discrete geometry [2,3,4,5]. The interpretation of antisymmetry as the two-colored symmetry brought the idea of Belov multi-colored symmetry [6], named in [4,5] the p-symmetry. As the other generalization of antisymmetry appeared Zamorzaev antisymmetry of diferent patterns (multiple antisymmetry [3], or (2,2,...,2)-symmetry where the number 2 is repeated l times (or shortly (2^l)-symmetry), extending the antisymmetry by assigning to the points of a transformed figure not only one, but several qualitatively different signs + or -.

As the synthesis of the two generalizations of antisymmetry mentioned, resulted the notion of colored antisymmetry (the simple, or $(p,2)$-symmetry [4,5], as well as the multiple, or $(p,2,...,2)$-symmetry [5], $(p,2^l)$-symmetry).

Till this time, together with the two-dimensional groups and their subgroups, the three-dimensional point, line and layer $(p,2)$- and $(p,2^l)$-symmetry groups [4,5], and also partly the $(p,2)$-symmetry space groups $G_3^{1,p}$ [7], are studied. For completing the scheme of crystallographic $(p,2^l)$-symmetry groups we need only the $(p,2)$- and $(p,2^l)$-symmetry space groups $G_3^{1,p}$ and $G_3^{l,p}$.

The survey of the complete derivation of symmorphic $(p,2^l)$-symmetry space groups $G_3^{l,p}$ for all nontrivial cases of assigning to points of three-dimensional Euclidean space colored by p colors and l signs + or -, is the purpose of this work.

1. Basic assumptions of the general theory of colored antisymmetry of different patterns

§1. The colored antisymmetry of different patterns is defined as follows:

a) to the every point of a figure, comprising the one of p colors ($p \geq 3$), the l signs + or - are assigned;

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b) by the transformation of the colored antisymmetry of the 0- pattern, j-pattern, (j,k)-pattern,..., or $(1,2,...,l)$-pattern is called the isometric transformation of the "indexed" figure considered, transforming a point with the color i into the point with the color $(i+m)$ (or with color $(i+m-p)$, where m is the fixed for all $i = 1,2,...,p$); consequently, such a transformation does not change any sign, changes only jth sign, only jth and kth,..., or all l signs. Altogether they are 2^l patterns of colored antisymmetry, where the transformations of colored antisymmetry of the 0-pattern coincide with the colored symmetries. Therefore, for $p \geq 3$ and $l = 0$ this definition results in the p-symmetry [4], for $p = 0$ and $l \geq 2$ in the l-multiple antisymmetry [3], and for $p \geq 3$ and $l = 1$ in $(p,2)$-symmetry (the colored antisymmetry of Neronova-Belov [4]).

It is simply to check, that all the transformations of colored antisymmetry of different patterns of a given figure consist of the group called the $(p,2^l)$-symmetry group of this figure.

From the above follows that every transformation of the colored antisymmetry of a certain pattern $g = \varepsilon s$, where s is a symmetry transformation, and ε is a permutation of "indexes" from the group $P = \{(1,2,...,p)\times(+,-)\times...\times(+,-)\} \equiv C_p \times C_2^{(1)} \times ... \times C_2^{(l)} = C_p \times C_2^l$, where C_p and $C_2^{(i)}$ are, respectively, cyclic groups of order p and order 2, C_2 - the group of order 2 , and $\times$ denotes the direct product of groups.

The $(p,2^l)$-symmetry group $G^{l,p}$ is called the group of complete $(p,2^l)$-symmetry if the group P_1 consisting of l-components of the transformations belonging to $G^{l,p}$, coincides with P. If P_1 is a nontrivial subgroup of P, the $G^{l,p}$ is called the group of uncomplete $(p,2^l)$-symmetry [4,5].

The method of derivation of $(p,2^l)$-symmetry groups of a certain category from already found groups of p-symmetry from the same category is given by the following theorem:

THEOREM 1. *Every group of $(p,2^l)$-symmetry, not containing $(p,2^l)$-identity transformations* ,can be derived from a certain junior group of p-symmetry (p-generating) using one of the following ways:*

a) *by extending the p-generating antiidentity transformations of k independent patterns $(1 \leq k \leq l)$** ,which generate the group C_2^k of the order 2^k and commute with the elements of the first; the group obtained is their direct product and is called the p-senior group of k independent (2^k-1 different) patterns (the group $p-S^k$);*

b) *by replacing all elements in each coset in its decomposition according to one of normal subgroups of index 2^m (which is the section of 2^m-1 different subgroups of the index 2 by the corresponding transformations of the colored antisymmetry of any from 2^m-1 dependent patterns, among which they are m independent (2^m-1 different) patterns (group $p-M^m$);*

c) *by extending the group obtained from it using the method* (b) *by such a system*

* This means, $(p,2^l)$-symmetry transformations keeping points of the figure fixed, and changing only indexes [4,5].

** The independence of colored symmetry patterns is defined as the independence of the corresponding antisymmetry patterns [4,5].

of antiidentities (a), *forming together with already existing colored antisymmetry patterns $k+m$ independent patterns of colored antisymmetry; the result is the p-senior group of k independent (2^k-1 different) patterns and junior of m independent ($2^{k+m}-2^k$ different patterns ($1 \leq k, m$; $2 \leq k+m \leq l$ the group $p - S^k M^m$); this group is the direct product of the group $p - M^m$ with the group of the order 2^k, generated by the antiidentities mentioned.*

PROOF: Let $G^{l,p}$ be a group of $(p, 2^l)$-symmetry, not containing $(p, 2^l)$-identity transformations. In the most general case such a group consists of symmetry transformations s, p-symmetries $s\varepsilon$ (where ε is non-trivial p-identity*), p-antisymmetries $s\varepsilon\varepsilon'$ (where ε' is the antiidentity transformation of a certain pattern) and antisymmetry transformations $s\varepsilon'$. Let define the homomorphism of a group $G^{l,p}$ into a group $E^{(l)}$, which is the direct product of l groups of the order 2 generated by antiidentities of j-pattern ($j = 1, 2, ...,$) as:

$$h(g) = \begin{cases} e; & \text{if } g = s \text{ or } g = s\varepsilon \\ \varepsilon'; & \text{if } g = s\varepsilon\varepsilon' \text{ or } g = s\varepsilon'. \end{cases}$$

It is easy to prove that h is a homomorphism with the carnel G, where G is a subgroup of $G^{l,p}$ consisting of all symmetry and p-symmetry transformations ($G = G^{l,p} \cap G^p$, where G^p is the p-generating group for $G^{l,p}$). In line with homomorphism property [8], $G^{l,p}/G \equiv E^{(l)}$, and complete original of each subgroup E_1 of $E^{(l)}$ is the subgroup G_1 of $G^{l,p}$, where $G^{l,p}/G_1 \equiv E^{(l)}/E_1$, and E_1 is the multiple antisymmetry group of k independent patterns.

From its side, the transformation

$$h'(g) = \begin{cases} s; & \text{if } g = s \text{ or } g = s\varepsilon' \\ s\varepsilon; & \text{if } g = s\varepsilon \text{ or } g = s\varepsilon\varepsilon'. \end{cases}$$

defines the homomorphism h' of the group $G^{l,p}$ into the set G^p of l-antisymmetry transformations, corresponding to the transformations of $G^{l,p}$; according to the property of homomorphism, the set mentioned is the group, and the carnel of homomorphism h' is the subgroup $E = G^{l,p} \cap E^{(l)}$.

If the group $G^{l,p}$ is of the type $p - S^k$, then $h(G^{l,p}) = E$ and $G^p \subset G^{l,p}$, i.e. G^p coincides with G and represents the carnel of the homomorphism h, transforming $G^{l,p}$ on E, so $G^{l,p} = G^p \times E$.

When $G^{l,p}$ is of the type $p - M^m$, because $E = e$, h is isomorphism, and, because it is the section of $2^m - 1$ subgroups of the index 2, the symmetry subgroup $S = G^{l,p} \cap G^p$ is of the same index 2^m in $G^{l,p}$ and G^p.

Finaly, if $G^{l,p}$ is of the type $p - S^k M^m$, then $h'(G) = E' = E_1 \times E$, where E_1 is generated by m independent antiidentities; then $G_1 = h'^{-1}(E_1)$ is the subgroup of the index 2^k in $G^{l,p}$ (because $G^{l,p}/G_1 \equiv E'/E_1 \equiv E$) and of the type p-junior of m independent patterns, making together with k patterns present in E the system

* This means, it is not the identical permutation of indexes $1, 2, 3, ..., p$ (see [4], page 36.

of $k+m$ independent patterns. Therefore, according to Chapter II, §2 of the monograph [3], $G^{l,p} = G_1 \times E$. ■

Also holds the converse statement: the set $G^{l,p}$, obtained from a certain junior p-symmetry group G^p by the method a), b), or c) of Theorem 1, is the group.

The collection of the groups derived from one p-generating group by all methods given by Theorem 1 is called the $(p, 2^l)$-family. From the theorem and its proof result the general properties of a $(p, 2^l)$-family. For every $(p, 2^l)$-family: a) all groups of the type $p - M^m$ $(1 \leq m \leq l)$ are isomorphic to their p-generating group; b) for every fixed k $(1 \leq k \leq l)$ all groups of the type $p - S^k M^m$ $(0 \leq m \leq l-k)$ are mutually isomorphic.

In each transition from $l-1$ to l signs only nontrivial is the derivation of groups of the type $p - M^m$. Practically, such groups can be efficiently derived using the method of Shubnikov-Zamorzaev [1,3]: l or more transformations in the system of generators of the p-generating group are replaced by the corresponding transformations of colored antisymmetry of different patterns, among which they are l independent. Among the groups of $(p, 2^l)$-symmetry obtained, in each family we must find the equal ones*) and exclude the groups containing $(p, 2^l)$-identity transformations.

Using this method, in order to obtain from a p-generating group the groups of the type $p - M^m$ non containing $(p, 2^l)$-identity transformations, we must respect the following rules:

a) a generating element of p-generating group can be replaced by a colored antisymmetry transformation *iff* its corresponding element in original classical-symmetry group (generating for the p-junior group in question) has been replaced by antisymmetry transformation in the derivation of junior Shubnikov groups;

b) two generating elements of a p-generating group is possible to replace by colored antisymmetry transformations of two different patterns, or replace separately by colored antisymmetry transformations *iff* their corresponding elements of the generating classical-symmetry group have been replace separately in the derivation of junior Shubnikov groups;

c) three generating elements of a p-generating group can be replaced all by colored antisymmetry transformations of three independent patterns, or in pairs by colored antisymmetry transformations of different patterns, or separately *iff* their corresponding elements of the generating classical-symmetry have been replaced separately in the derivation of junior Shubnikov groups, etc.

Validity of these principles is evident. In fact, because of the possible decomposibility of $(p, 2^l)$-symmetry in p- and 2^l-symmetry if the indexes (colors) are neglected, the results are statements of the theory of l-multiple antisymmetry, giving the methods of derivation of Zamorzaev groups from the classical, presented in Chapter II, §2 of the monograph [3].

* Two groups G and G' of the same category $G_r^{l,p}$ are equal if there is an affine transformation of r-dimensional space transforming P-affine (this means, by permuting indexes from the group P) every class of points, transformed by G onto the class of points, transformed by G' [4].

For p - an even number, in extension from a p-generating group to the group of the type $p-M^m$ using Shubnikov-Zamorzaev method, they are derived the groups of complete and uncomplete $(p,2^l)$-symmetry. To find the structure of the derived group $G^{l,p}$ it can be substituted by the combination of groups of p-symmetry and simple antisymmetry $G^p, G^1, ..., G^l$ ($G^{l,p} \equiv (G^p, G^1, ..., G^l)$ (where G^p is obtained from $G^{l,p}$ by ignoring changes of signs, and G^i are obtained from $G^{l,p}$ by ignoring the indexes and $l-1$ sign of transformed "signed" figure and keeping only i th sign), and forming the extended symbol $S/(H_0, H_1, ..., H_l)/H$, where $H_0, H_1, ..., H_l$ are, respectively, the symmetry subgroups of G^p, $H_1, ..., H_l$, and $H = H_0 \cap H_1 ... \cap H_l$. If, for some $i = 1, 2, ..., l$, $H = H_0 \cap H_i$, then $G^{l,p}$ is the group of uncomplete $(p,2^l)$-symmetry. If for all such i holds $H \neq H_0 \cap H_i$, then $G^{l,p}$ is the group of complete $(p,2^l)$-symmetry [4].

§2. The problem of the derivation of junior $(p,2^l)$-symmetry groups $G^{l,p}$ directly from the p-generating groups can be solved very efficiently using their antisymmetric characteristics (AC). We are giving the short theoretical background of the AC-method.

Let all products of the generators of discrete symmetry group G be formed in this way, that in each product every generating element appears mostly once. After that, we divide the set obtained on the subsets of transformations equivalent in the sense of symmetry (this means, having the same role in the symmetry group G). The system obtained is called the antisymmetry characteristic of the group G.

If G^p is a junior p-symmetry group obtained from a generating group G using the method [4] (i.e., replacing its generators by the corresponding p-symmetry transformations), we treat the generators of the group G^p in the same way as it is done with the generators of its generating symmetry group G. The result obtained is the antisymmetry characteristic of the group $G^p(AC(G^p))$.

The transition from G to G^p induces the transition from $AC(G)$ to $AC(G^p)$, which will be used as the basis for the derivation of $(p,2^l)$-symmetry groups of the type $p-M^m$ from G^p .

The structure of the permutation group $P \equiv C_p \times C_2^l$ depends on the reducibility of the group C_p (i.e. from the possibility that C_p be decomposed in the direct product of groups $C^{p'}$, and C_2). If $p = 2n-1$ or $4n$, such a decomposition is imposible. If $p = 4n-2$, then $C_{4n-2} = C_{2n-1} \times C_2$, and consequently, the $(2n-1,2)$-symmetry is equivalent to $(4n-2, 2^{l-1})$-symmetry. It is easy to check [4,5] that the derivation of the groups of $(2n-1, 2^l)$-symmetry of the type $p-M^m$ from a p-generating group G^{2n-1} is the identical to the derivation of 2^l-symmetry groups of the M^m-type from a symmetry group G from which G^{2n-1} is derived, and the derivation of $(4n-2, 2^{l-1})$-symmetry groups of the type $p-M^m$ from a p-generating group G^{4n-2} is identical to the derivation of $(2n-1, 2^l)$-symmetry groups of the type $p-M^m$ from the $p-M^1$ group $G^{1,2n-1}$ or G^1, derived from the same symmetry group.

Since we consider only the case $p = 3, 4, 6$, the number of the groups of $(3, 2^l)$-symmetry of the type $p-M^m$ derived from a p-generating group G^3, is equal to the number of the groups of 2^l-symmetry (i.e. l-multiple antisymmetry) of the

M^m-type derived from the same generating symmetry group, and the number of $(6, 2^{l-1})$-symmetry groups of the type $p - M^m$ derived from the p-generating group G^6 is equal to the number of $(3, 2^l)$-symmetry groups of the type $p - M^m$, derived from the group of (3,2)-symmetry $G^{1,3}$, or antisymmetry G^1, belonging to the same family with G^6. The number of $(4, 2^l)$-symmetry groups of the type $p - M^m$ derived from a p-generating group G^4 it is not possible to find directly, but it can be the same or larger then the number of 2^l-symmetry groups of the M^m -type derived from the classical-symmetry group G rising the family of G^4 [4].

The proposed method for the derivation of junior $(p, 2^l)$-symmetry groups directly from the generating p-symmetry group G^p makes possible to find the groups of complete and uncomplete $(p, 2^l)$-symmetry. Let the symmetry group G be given by its presentation, this means by the set of generators $\{S_1, S_2, ..., S_k\}$ and their relations $g_n(S_1, S_2, ..., S_k) = E$, $n = 1, 2, ...t$, and let the permutation group $P = C_p \times C_2^l$ be given by its presentation $\{e_1, e_2, ..., e_{l+1}\}$, $e_1^p = E$, $e_i^2 = E$, $e_ie_j = e_je_i$, $i = 2, 3, ..., l+1$, $j = 1, 2, ..., l+1$, and $S_qe_j = e_jS_q$, $q = 1, 2, ..., k$.

Let us consider $(p, 2^l)$-symmetry groups $G^{l,p}$ derived from G According to the general theory of $(p, 2^l)$-symmetry given above, we conclude that:

a) the junior among them are the groups keeping satisfied all relations from the presentation of G after replacing the generators by the corresponding $(p, 2^l)$-symmetry generators;

b) a junior $(p, 2^l)$-symmetry group is of the type $p - M^m$, if it remains the l-multiple antisymmetry group of the M^m-type after neglecting the indexes $1, 2, ..., p$, but preserving the collection of l signs + or -;

c) a junior group of the type $p - M^m$ is the group of complete $(p, 2^l)$-symmetry if from the collection of permutations and sign-changes corresponding to its transformations it is possible to set apart a system of generators of the group $P = C_p \times C_2^l$. If the contrary, such a group of the type $p - M^m$ is the uncomplete $(p, 2^l)$-symmetry group.

For $p = 2n - 1$ the generator e_1 has no influence on the validity of the condition c), because $e_1^p = e_1^{2n-1} = E$ implies $e^{2n} = e_1$, but for $p = 2n$, $e_1^{2n} = E$. Therefore, checking the condition c) for some group $G^{l,p}$ for the p - an even number it is sufficient to consider the powers of the generator e_1 of the group C_p according *mod 2*, by using the homomorphism of C_p into $C_2 : e_1^{2n-1} \to e_1$, $e_1^{2n} \to E$.

So, the group $G^{l,p}$ of the type $p - M^l$, for the p - an odd number is the group of complete $(p, 2^l)$-symmetry if when neglecting the indexes $1, 2, ..., p$ it turns into the M^m -type group, and for the p - an even number, if the group G^{l+1} obtained from it by using the homomorphism mentioned, is the M^{l+1}-type group.

Because of the equivalence of $(2n - 1, 2)$ - and $(4n - 2, 2^{l-1})$-symmetry, for the numbers $N_m^{(p)}$ of all $(p, 2^l)$-symmetry groups of the type $p - M^m$ and the numbers $(N_m^{(p)})$ of the groups of the type $p - M^m$ of the complete $(p, 2^l)$-symmetry, where the groups in question belong to the same category, hold the following relationships:

$N_m^{(2n-1)} = (2^m - 1)N_{m-1}^{(2n-1)} + N_m^{(4n-2)} = (N_m^{(4n-2)})$ for p - an even number, and for p - an odd number $(N_m^{(p)}) = N_m^{(p)} - (2m - 1)(N_{m-1}^{(p)})$, supposing that $N_0^{(p)} = 1$.

2. Survey of complete derivation of junior $(p, 2^l)$-symmetry symmorphic space groups

§1. The $(p, 2^l)$-symmetry group $G^{l,p}$ of the threedimensional Euclidean space is called the space group of l-multiple colored antisymmetry if its $(p, 2^l)$-symmetry transformations satisfy the condition of space homogeneity, and its symmetries the condition of local discreteness (see the definition of Zamorzaev and Belov groups [3,4]).

Even for $p = 3, 4, 6$, the derivation of all space $(p, 2^l)$-symmetry groups of the type $p - M^m$ from the p -generating groups is enormously large. Therefore this problem we will divide on three parts: the derivation of junior symmorphic, hemisymorphic and asymmorphic $(p, 2^l)$-symmetry space groups.

In this paper is given the solution of the first part, and the other two will be considered in the proceeding work. In order to find the groups mentioned it is sufficient to extend all p-generating symmorphic groups from the second part of Appendix [4] to $(p, 2^l)$-symmetry groups. According to the general theory of $(p, 2^l)$-symmetry, using the methods given in Chapter 1, by such extension from 316 p-generating Belov symmorphic groups, the symmorphic $(p, 2^l)$-symmetry groups $G_3^{l,p}$ will be derived. Since each p-generating group G_3^p is given by a finite system of p-generators, for given p and l the number of different replacements of p-generators by $(p, 2^l)$-symmetry generators containing l independent, is also finite, and, consequently, the number of all junior space groups $G^{l,p}$ in each family is finite.

For the space groups of l-multiple colored antisymmetry $G_3^{l,p}$ the number of all different groups of each fixed type $p - M^m$ is $N_m^{(p)}$, the same is the number of different groups of the type $p - S^k M^m$ for $1 \leq k \leq l - m$, and for the type $p - S^k$ this number is equal to the number p-generating groups $N^{(p)}$ (i.e. the junior groups of the category G_3^p). Hence, in the transition from $l - 1$ to l signs, nontrivial is only the account of the groups $p - M^l$; but, for sufficiently large values of l ($l \geq 6$ for the symmorphic groups $G_3^{l,p}$) $N_l^{(p)} = 0$.

As a final conclusion, for all symmorphic $(p, 2^l)$-symmetry space groups $G_3^{l,p}$ of the type $p - M^m$ ($p = 3, 4, 6$), the numbers $N_m^{(p)}$ are the following: $N_1^{(3)} = 165$, $N_2^{(3)} = 1038$, $N_3^{(3)} = 10473$, $N_4^{(3)} = 126000$, $N_5^{(3)} = 1249920$, and $N_l^{(3)} = 0$ for $l \geq 6$; $N_1^{(4)} = 779$, $N_2^{(4)} = 8278$, $N_3^{(4)} = 1127633$, $N_4^{(4)} = 1680000$, $N_5^{(4)} = 19998720$, and $N_l^{(4)} = 0$ for $l \geq 6$; $N_1^{(6)} = 1203$, $N_2^{(6)} = 13587$, $N_3^{(6)} = 199311$, $N_4^{(6)} = 3139920$, $N_5^{(6)} = 38747520$, and $N_l^{(6)} = 0$ for $l \geq 6$.

The corresponding numbers $(N_m^{(p)})$ are: $(N_l^{(3)}) = N_l^{(3)}$; $(N_1^{(4)}) = 675$, $(N_2^{(4)}) = 6253$, $(N_3^{(4)}) = 68992$, $(N_4^{(4)}) = 645120$, and $N_l^{(4)} = 0$ for $l \geq 5$; $(N_1^{(6)}) = 1038$, $(N_2^{(6)}) = 10473$, $(N_3^{(6)}) = 126000$, $(N_4^{(6)}) = 1249920$, and $N_l^{(6)} = 0$ for $l \geq 5$.

§2. In order to explain the presented final results we are giving the example of their derivation and the computation of the numbers mentioned.

Translation space group $1s$ (or $P1$) given by the system of generators $\{a, b, c\}$, with the antisymmetric characteristic AC : $\{a, b, c, ab, ac, bc, abc\}$, according [10], generates three p-symmetry groups for $p = 3, 4, 6$: 1) $\{a^{(3)}, b, c\}$, 2) $\{a^{(4)}, b, c\}$, 3)

$\{a^{(6)}, b, c\}$ (Table P2 of the monograph [4]).

According to the theoretical bacground given in Chapter 1, §2, we conclude that AC of the group 1) remines the same as $AC(1s)$, so the derivation of $(3, 2^l)$-symmetry group from this 3-generating symmetry group of the type $3 - M^m$ is the same as the derivation of (2^l)-symmetry groups of the M^m-type from the classical-symmetry group $1s$. Hence, there is one (3,2)-symmetry group of the type $3 - M^1$ $\{\underline{a}^{(3)}, b, c\}$, one group of $(3, 2^2)$-symmetry of the type $3 - M^2$ $\{\underline{a}^{(3)}, b', c\}$, one group of $(3, 2^3)$-symmetry of the type $3 - M^3$ $\{\underline{a}^{(3)}, b', *c\}$, and no groups of $(3, 2^l)$-symmetry of type $3 - M^l$ for $l \geq 4$.

The transition from the generating symmetry group 1s to 4-junior group 2) induces the transition from $AC(\{a, b, c\}) = \{a, b, c, ab, ac, bc, abc\}$ to $AC(\{a^{(4)}, b, c) = \{e_1a, b, c, e_1ab, e_1ac, bc, e_1abc\}$ which falls in two subsets of transformations equivalent in the sense of 4-symmetry $\{e_1a, e_1ab, e_1ac, e_1abc\}$ and $\{b, c, bc\}$ forming the $AC(\{a^{(4)}, b, c\}) = \{a, ab, ac, abc\}\{b, c, bc\}$, the reduced form of which is $\{a, ab, ac, abc\}$. According to [10], this reduced $AC = \{a, ab, ac, abc\}$ belongs to the AC-equivalence class $XXXI$, represented by the group $1a$ [10,11]; therefore, the group $\{a^{(4)}, b, c\}$ gives the same number of the $(4, 2^l)$-symmetry groups of the type $4 - M^m$, as the group $1a$ (2^l)-symmetry groups of the M^m-type.

Hence, the 4-colored group 2) generates two groups of (4,2)-symmetry of the type $4 - M^1$ $\{a^{(4)}, \underline{b}, c\}$, $\{\underline{a}^{(4)}, b, c\}$. The first is the group of complete, and the second of uncomplete (4,2)-symmetry. According Chapter 1, §1, this can be concluded from their extended symbols $\{a, b, c\}/\{4a, b, c\}$, $\{a, 2b, c\}/\{4a, b, c\}$ and $\{a, b, c\}/\{4a, b, c\}$, $\{2a, b, c\}/\{4a, b, c\}$. In the first case the symmetry subgroup $\{4a, b, c\}$ coincides to the section of the symmetry subgroups of the groups $\{a^{(4)}, b, c\}$ and $\{a, \underline{b}, c\}$ on which it splits, and in the second its symmetry subgroup $\{4a, b, c\}$ coincides to the section of the symmetry subgroups of the groups $\{a^{(4)}, b, c\}$ and $\{\underline{a}, b, c\}$, defining together the group $\{\underline{a}^{(4)}, b, c\}$. Proceeding in the same manner, from 4-generating group $\{a^{(4)}, b, c\}$ we derive four groups of $(4, 2^2)$-symmetry of the type $4 - M^2$ $\{a^{(4)}, \underline{b}, c'\}$, $\{\underline{a}^{(4)}, b', c\}$, $\{a^{(4)\prime}, \underline{b}, c\}$ and $\{\underline{a}^{(4)\prime}, \underline{b}, c\}$ from which, as we can conclude from their extended symbols, only the first is the complete $(4, 2^2)$-symmetry group. Finaly, from the same 4-generating group we derive seven groups of the uncomplete $(4, 2^3)$-symmetry of the type $4 - M^3$ $\{\underline{a}^{(4)}, b', *c'\}$, $\{a^{(4)\prime}, *b, \underline{c}\}$, $\{*a^{(4)\prime}, \underline{b}, c'\}$, $\{\underline{a}^{(4)}, *b', c'\}$, $\{a^{(4)\prime}, *\underline{b}, *c\}$, $\{*a^{(4)}, \underline{b}', \underline{c}\}$ and $\{\underline{a}^{(4)\prime}, *\underline{b}, \underline{c}\}$, and no groups of $(4, 2^l)$-symmetry of the type $4 - M^l$ for $l \geq 4$.

The acount of the derivation of complete $(4, 2^l)$-symmetry groups of the type $4 - M^m$ from the 4-generating group $\{a^{(4)}, b, c\}$ it is possible to obtain without making the complete catalogue. Using the homomorphism $C_4 \to C_2$ given in Chapter 2, §1, we can find that $AC(\{a^{(4)}, b, c\}$ is of the form $\{e_1, e_1, e_1, e_1\}$ type $(5)^1$. Using the results [11, Appendix], we can simply calculate the numbers $N_m(\{a^{(4)}, b, c\})$ and $(N_m(\{a^{(4)}, b, c\}))$.

Treating in the same way the 6-generating group $\{a^{(6)}, b, c\}$ we obtain the same results as for $\{a^{(4)}, b, c\}$. Because the p-generating groups $\{a^{(4)}, b, c\}$ and $\{a^{(6)}, b, c\}$ belong to the same AC-equivalence class, for every fixed m they will give the same numerical results, and even more, the $(p, 2^l)$-symmetry groups obtained will be corresponding in pairs with regard to their structure (this means, one $(4, 2^l)$-

symmetry group of the type $4-M^m$ can be transformed in the corresponding $(6,2^l)$-symmetry group of the type $6-M^m$ replacing the index 4 by 6, and keeping unchanged the set of symbols _, ', * and their combinations denoting the multiple antisymmetry transformations.

Because the complete cataloguation of all symmorphic $(p,2^l)$-symmetry group $G_3^{l,p}$ is to large even for their simplest type $p-M^1$, we will give only the numerical results and the acount of these groups, obtained using the AC-method.

The survey of all p-generating symmorphic space groups G_3^p $(p=3,4,6)$ is classified in the families [4, P2] and followed by their AC and by the number of the AC-equivalence class [10,11]. The existential conditions remine the same. Wishing to make his own control of the final results, the reader can simply find the form and the type of each $AC(G^{(p)}))$, and using the results [11, Appendix] calculate the numbers $N_m(G^{(p)})$ and $(N_m(G^{(p)}))$.

Table 1

$I)$ $1s, P1, \{a,b,c\}$,	$AC: \{a,b,c,ab,ac,bc,abc\}$;
1) $\{a^{(3)},b,c\}$,	$AC: \{a,b,c,ab,ac,bc,abc\}$;
2) $\{a^{(4)},b,c\}$,	$AC: \{a,ab,ac,abc\}$, $XXXI, 1a$;
3) $\{a^{(6)},b,c\}$,	$AC: \{a,ab,ac,abc\}$, $XXXI, 1a$.
$III)$ $3s, P2, \{a,b,c\}(2)$,	$AC: \{c\}\{2,2a,2b,2ab\}$;
1) $\{a,b,c^{(3)}\}(2)$,	$AC: \{c\}\{2,2a,2b,2ab\}$;
2) $\{a,b,c^{(4)}\}(2)$,	$AC: \{c\}\{2,2a,2b,2ab\}$;
3) $\{a,b,c^{(4)}\}(2^{(2)})$,	$AC: \{c\}\{2,2a,2b,2ab\}$;
4) $\{a^{(2)},b,c^{(4)}\}(2)$,	$AC: \{c\}\{2,2b\}\{2a,2ab\}$, $VIII, 8s$;
5) $\{a,b,c^{(3)}\}(2^{(2)})$,	$AC: \{c\}\{2,2a,2b,2ab\}$;
6) $\{a,b,c^{(6)}\}(2)$,	$AC: \{c\}\{2,2a,2b,2ab\}$;
7) $\{a^{(2)},b,c^{(3)}\}(2)$,	$AC: \{c\}\{2,2b\}\{2a,2ab\}$, $VIII, 8s$;
8) $\{a,b,c^{(6)}\}(2^{(2)})$,	$AC: \{c\}\{2,2a,2b,2ab\}$;
9) $\{a^{(2)},b,c^{(6)}\}(2)$,	$AC: \{c\}\{2,2b\}\{2a,2ab\}$, $VIII, 8s$.
$IV)$ $4s, B2, \{a,b,(a+c)/2\}(2)$,	$AC: \{2,2b\}\{2(a+c)/2, 2b(a+c)/2\}$;
1) $\{a,b,(a+c)/2^{(3)}\}(2)$,	$AC: \{2,2b\}\{2(a+c)/2, 2b(a+c)/2\}$;
2) $\{a,b,(a+c)/2^{(4)}\}(2)$,	$AC: \{2,2b\}\{2(a+c)/2, 2b(a+c)/2\}$;
3) $\{a,b,(a+c)/2^{(4)}\}(2^{(2)})$,	$AC: \{2,2b\}\{2(a+c)/2, 2b(a+c)/2\}$;
4) $\{a,b^{(2)},(a+c)/2^{(4)}\}(2)$,	$AC: \{2\}\{2(a+c)/2, 2b(a+c)/2\}$, $VI, 6s$;
5) $\{a,b,(a+c)/2^{(3)}\}(2^{(2)})$,	$AC: \{2,2b\}\{2(a+c)/2, 2b(a+c)/2\}$;
6) $\{a,b,(a+c)/2^{(6)}\}(2)$,	$AC: \{2,2b\}\{2(a+c)/2, 2b(a+c)/2\}$;
7) $\{a,b,(a+c)/2^{(6)}\}(2^{(2)})$,	$AC: \{2,2b\}\{2(a+c)/2, 2b(a+c)/2\}$;
8) $\{a,b^{(2)},(a+c)/2^{(3)}\}(2)$,	$AC: \{2\}\{b\}\{(a+c)/2\}$, $XX, 25s$.
$V)$ $5s, Pm, \{a,b,c\}(m)$,	$AC: \{a,b,ab\}\{m,mc\}$;
1) $\{a^{(3)},b,c\}(m)$,	$AC: \{a,b,ab\}\{m,mc\}$;
2) $\{a^{(4)},b,c\}(m)$,	$AC: \{a,ab\}\{m,mc\}$, $XXII, 28s$;
3) $\{a^{(4)},b,c\}(m^{(2)})$,	$AC: \{a,ab\}\{m,mc\}$, $XXII, 28s$;
4) $\{a^{(4)},b,c^{(2)}\}(m)$,	$AC: \{a,ab\}\{m\}\{c\}$, $XIII, 14s$;
5) $\{a^{(3)},b,c\}(m^{(2)})$,	$AC: \{a,b,ab\}\{m,mc\}$;
6) $\{a^{(6)},b,c\}(m)$,	$AC: \{a,ab\}\{m,mc\}$, $XXII, 28s$;

7) $\{a^{(3)}, b, c^{(2)}\}(m)$, $AC: \{c\}\{m\}\{a, b, ab\}, XVII, 20s$;
8) $\{a^{(6)}, b, c\}(m^{(2)})$, $AC: \{a, ab\}\{m, mc\}, XXII, 28s$;
9) $\{a^{(6)}, b, c^{(2)}\}(m)$, $AC: \{a, ab\}\{m\}\{c\}, XIII, 14s$.
$VI)$ $6s, Bm, \{a, b, (a+c)/2\}(m)$, $AC: \{m\}\{(a+c)/2, b(a+c)/2\}$;
1) $\{a^{(3)}, b, (a+c)/2\}(m)$, $AC: \{m\}\{(a+c)/2, b(a+c)/2\}$;
2) $\{a, b^{(4)}, (a+c)/2\}(m)$, $AC: \{m\}\{(a+c)/2\}\{b\}, XX, 25s$;
3) $\{a, b^{(4)}, (a+c)/2\}(m^{(2)})$, $AC: \{m\}\{(a+c)/2\}\{b\}, XX, 25s$;
4) $\{a^{(2)}, b, (a+c)/2^{(4)}\}(m)$, $AC: \{m\}\{(a+c)/2, b(a+c)/2\}$;
5) $\{a^{(2)}, b, (a+c)/2^{(4)}\}(m^{(2)})$, $AC: \{m\}\{(a+c)/2, b(a+c)/2\}$;
6) $\{a, b^{(3)}, (a+c)/2\}(m^{(2)})$, $AC: \{m\}\{(a+c)/2, b(a+c)/2\}$;
7) $\{a, b^{(6)}, (a+c)/2\}(m)$, $AC: \{m\}\{(a+c)/2\}\{b\}, XX, 25s$;
8) $\{a, b^{(6)}, (a+c)/2\}(m^{(2)})$, $AC: \{m\}\{(a+c)/2\}\{b\}, XX, 25s$;
9) $\{a, b^{(3)}, (a+c)/2^{(2)}\}(m)$, $AC: \{m\}\{(a+c)/2, b(a+c)/2\}$;
10) $\{a, b^{(3)}, (a+c)/2^{(2)}\}(m^{(2)}\}$, $AC: \{m\}\{(a+c)/2, b(a+c)/2\}$.
$X)$ $11s, I222, \{a, b, (a+b+c)/2\}(2:2'), AC: \{(a+b+c)/2\}\{2, 2', 22'\}$;
1) $\{a^{(2)}, b^{(2)}, (a+b+c)/2^{(4)}\}(2:2')$, $AC: \{(a+b+c)/2\}\{2, 2', 22'\}$.
$XII)$ $13s, Pmm2, \{a, b, c\}(2m)$, $AC: \{c\}\{\{m, ma\}, \{2m, 2mb\}\}$;
1) $\{a, b, c^{(3)}\}(2m)$, $AC: \{c\}\{\{m, ma\}, \{2m, 2mb\}\}$;
2) $\{a, b, c^{(4)}\}(2m)$, $AC: \{c\}\{\{m, ma\}, \{2m, 2mb\}\}$;
3) $\{a, b, c^{(4)}\}(2m^{(2)})$, $AC: \{c\}\{\{m, ma\}, \{2m, 2mb\}\}$;
4) $\{a, b, c^{(4)}\}(2^{(2)}m)$, $AC: \{c\}\{m, ma\}\{2m, 2mb\}, XVI, 19s$;
5) $\{a^{(2)}, b, c^{(4)}\}(2m)$, $AC: \{a\}\{c\}\{m\}\{2m, 2mb\}, 13s.5$;
6) $\{a^{(2)}, b, c^{(4)}\}(2m^{(2)})$, $AC: \{a\}\{c\}\{m\}\{2m, 2mb\}, 13s.5$
7) $\{a^{(2)}, b^{(2)}, c^{(4)}\}(2m)$, $AC: \{c\}\{/m, ma/, /2m, 2mb/\}, 13.7*$;
8) $\{a, b, c^{(3)}\}(2m^{(2)})$, $AC: \{c\}\{\{m, ma\}, \{2m, 2mb\}\}$;
9) $\{a, b, c^{(3)}\}(2^{(2)}m)$, $AC: \{c\}\{m, ma\}\{2m, 2mb\}, XVI, 19s$;
10) $\{a, b, c^{(6)}\}(2m)$, $AC: \{c\}\{\{m, ma\}, \{2m, 2mb\}\}$;
11) $\{a, b, c^{(6)}\}(2m^{(2)})$, $AC: \{c\}\{\{m, ma\}, \{2m, 2mb\}\}$;
12) $\{a, b, c^{(6)}\}(2^{(2)}m)$, $AC: \{c\}\{m, ma\}\{2m, 2mb\}, XVI, 19s$;
13) $\{a^{(2)}, b, c^{(3)}\}(2m)$, $AC: \{a\}\{c\}\{m\}\{2m, 2mb\}, 13s.5$;
14) $\{a^{(2)}, b, c^{(3)}\}(2m^{(2)})$, $AC: \{a\}\{c\}\{m\}\{2m, 2mb\}, 13s.5$;
15) $\{a^{(2)}, b, c^{(6)}\}(2m)$, $AC: \{a\}\{c\}\{m\}\{2m, 2mb\}, 13s.5$;
16) $\{a^{(2)}, b, c^{(6)}\}(2m^{(2)})$, $AC: \{a\}\{c\}\{m\}\{2m, 2mb\}, 13s.5$;
17) $\{a^{(2)}, b^{(2)}, c^{(3)}\}(2m)$, $AC: \{c\}\{/m, ma/, /2m, 2mb/\}, 13s.7$;
18) $\{a^{(2)}, b^{(2)}, c^{(6)}\}(2m)$, $AC: \{c\}\{/m, ma/, /2m, 2mb/\}, 13s.7$.
$XIII)$ $14s, Cmm2, \{a, (a+b)/2, c\}(2m), AC: \{(a+b)/2\}\{c\}\{m, 2m\}$;
1) $\{a, (a+b)/2, c^{(3)}\}(2m)$, $AC: \{(a+b)/2\}\{c\}\{m, 2m\}$;
2) $\{a, (a+b)/2, c^{(4)}\}(2m)$, $AC: \{(a+b)/2\}\{c\}\{m, 2m\}$;
3) $\{a, (a+b)/2, c^{(4)}\}(2m^{(2)})$, $AC: \{(a+b)/2\}\{c\}\{m, 2m\}$;
4) $\{a, (a+b)/2, c^{(4)}\}(2^{(2)}m)$, $AC: \{(a+b)/2\}\{c\}\{m\}\{2\}, XXIII, 37s$;
5) $\{a, (a+b)/2^{(2)}, c^{(4)}\}(2m)$, $AC: \{(a+b)/2\}\{c\}\{m, 2m\}$;
6) $\{a, (a+b)/2^{(2)}, c^{(4)}\}(2m^{(2)})$, $AC: \{(a+b)/2\}\{c\}\{m, 2m\}$;

* The elements in the parentheses // remain fixed on their places [12]

7) $\{a,(a+b)/2^{(2)},c^{(4)}\}(2^{(2)}m)$, $AC:\{(a+b)/2\}\{c\}\{m\}\{2\}, XXIII, 37s$;
8) $\{a,(a+b)/2,c^{(3)}\}(2m^{(2)})$, $AC:\{(a+b)/2\}\{c\}\{m,2m\}$;
9) $\{a,(a+b)/2,c^{(3)}\}(2^{(2)}m)$, $AC:\{(a+b)/2\}\{c\}\{m\}\{2\}, XXIII, 37s$;
10) $\{a,(a+b)/2,c^{(6)}\}(2m)$, $AC:\{(a+b)/2\}\{c\}\{m,2m\}$;
11) $\{a,(a+b)/2,c^{(6)}\}(2m^{(2)})$, $AC:\{(a+b)/2\}\{c\}\{m,2m\}$;
12) $\{a,(a+b)/2,c^{(6)}\}(2^{(2)}m)$, $AC:\{(a+b)/2\}\{c\}\{m\}\{2\}, XXIII, 37s$;
13) $\{a,(a+b)/2^{(2)},c^{(3)}\}(2m)$, $AC:\{(a+b)/2\}\{c\}\{m,2m\}$;
14) $\{a,(a+b)/2^{(2)},c^{(3)}\}(2m^{(2)})$, $AC:\{(a+b)/2\}\{c\}\{m,2m\}$;
15) $\{a,(a+b)/2^{(2)},c^{(3)}\}(2^{(2)}m)$, $AC:\{(a+b)/2\}\{c\}\{m\}2\}, XXIII, 37s$;
16) $\{a,(a+b)/2^{(2)},c^{(6)}\}(2m)$, $AC:\{(a+b)/2\}\{c\}\{m,2m\}$;
17) $\{a,(a+b)/2^{(2)},c^{(6)}\}(2m^{(2)})$, $AC:\{(a+b)/2\}\{c\}\{m,2m\}$;
18) $\{a,(a+b)/2^{(2)},c^{(6)}\}(2^{(2)}m)$, $AC:\{(a+b)/2\}\{c\}\{m\}\{2\}, XXIII, 37s$.

$XIII)$ $15s, Bmm2, \{a,b,(a+c)/2\}(2m), AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
1) $\{a,b,(a+c)/2^{(3)}\}(2m)$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
2) $\{a,b,(a+c)/2^{(4)}\}(2m)$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
3) $\{a,b,(a+c)/2^{(4)}\}(2m^{(2)})$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
4) $\{a,b,(a+c)/2^{(4)}\}(2^{(2)}m)$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
5) $\{a,b,(a+c)/2^{(4)}\}(2^{(2)}m^{(2)})$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
6) $\{a,b^{(2)},(a+c)/2^{(4)}\}(2m)$, $AC:\{(a+c)/2\}\{m\}\{2\}\{b\}, XXIII, 37s$;
7) $\{a,b^{(2)},(a+c)/2^{(4)}\}(2m^{(2)})$, $AC:\{(a+c)/2\}\{m\}\{2\}\{b\}, XXIII, 37s$;
8) $\{a,b,(a+c)/2^{(3)}\}(2m^{(2)})$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
9) $\{a,b,(a+c)/2^{(3)}\}(2^{(2)}m)$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
10) $\{a,b,(a+c)/2^{(3)}\}(2^{(2)}m^{(2)})$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
11) $\{a,b,(a+c)/2^{(6)}\}(2m)$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
12) $\{a,b,(a+c)/2^{(6)}\}(2m^{(2)})$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
14) $\{a,b,(a+c)/2^{(6)}\}(2^{(2)}m^{(2)})$, $AC:\{(a+c)/2\}\{m\}\{2m,2mb\}$;
15) $\{a,b^{(2)},(a+c)/2^{(3)}\}(2m)$, $AC:\{(a+c)/2\}\{m\}\{2\}\{b\}, XXIII, 37s$;
16) $\{a,b^{(2)},(a+c)/2^{(3)}\}(2m^{(2)})$, $AC:\{(a+c)/2\}\{m\}\{2\}\{b\}, XXIII, 37s$;
17) $\{a,b^{(2)},(a+c)/2^{(6)}\}(2m)$, $AC:\{(a+c)/2\}\{m\}\{2\}\{b\}, XXIII, 37s$;
18) $\{a,b^{(2)},(a+c)/2^{(6)}\}(2m^{(2)})$, $AC:\{(a+c)/2\}\{m\}\{2\}\{b\}, XXIII, 37s$;

$VI)$ $16s, Imm2, \{a,b,(a+b+c)/2\}$ $(2m), AC:\{(a+b+c)/2\}\{m,2m\}$;
1) $\{a,b,(a+b+c)/2^{(3)}\}(2m)$, $AC:\{(a+b+c)/2\}\{m,2m\}$;
2) $\{a,b,(a+b+c)/2^{(4)}\}(2m)$, $AC:\{(a+b+c)/2\}\{m,2m\}$;
3) $\{a,b,(a+b+c)/2^{(4)}\}(2m^{(2)})$, $AC:\{(a+b+c)/2\}\{m,2m\}$;
4) $\{a,b,(a+b+c)/2^{(4)}\}(2^{(2)}m)$, $AC:\{(a+b+c)/2\}\{m\}\{2\}, XX, 25s$;
5) $\{a,b,(a+b+c)/2^{(3)}\}(2m^{(2)})$, $AC:\{(a+b+c)/2\}\{m,2m\}$;
6) $\{a,b,(a+b+c)/2^{(3)}\}(2^{(2)}m)$, $AC:\{(a+b+c)/2\}\{m\}\{2\}, XX, 25s$;
7) $\{a,b,(a+b+c)/2^{(6)}\}(2m)$, $AC:\{(a+b+c)/2\}\{m,2m\}$;
8) $\{a,b,(a+b+c)/2^{(6)}\}(2m^{(2)})$, $AC:\{(a+b+c)/2\}\{m,2m\}$;
9) $\{a,b,(a+b+c)/2^{(6)}\}(2^{(2)}m)$, $AC:\{(a+b+c)/2\}\{m\}\{2\}, XX, 25s$.

$XIV)$ $17s, Fmm2, \{a,(a+b)/2,(a+c)/2\}(2m)$,
$AC:\{(a+c)/2,(a+c)/2(a+b)/2\}\{m,2m\}\{m(a+c)/2,2m(a+c)/2(a+b)/2\}$;
1) $\{a,(a+b)/2,(a+c)/2(3)a\}(2m)$,
$AC:\{(a+c)/2,(a+c)/2(a+b)/2\}\{m,2m\}\{m(a+c)/2,2m(a+c)/2(a+b)/2\}$;

2) $\{a, (a+b)/2, (a+c)/2^{(4)}\}(2m)$,
$AC: \{(a+c)/2, (a+c)/2(a+b)/2\}\{m, 2m\}\{m(a+c)/2, 2m(a+c)/2(a+b)\}$;
3) $\{a, (a+b)/2, (a+c)/2^{(4)}\}(2m^{(2)})$,
$AC: \{(a+c)/2, (a+c)/2(a+b)/2\}\{m, 2m\}\{m(a+c)/2, 2m(a+c)/2(a+b)\}$;
4) $\{a, (a+b)/2, (a+c)/2(4)\}(2(2)m)$,
$AC: \{(a+c)/2\}\{(a+b)/2\}\{2\}\{m\}, XXIII, 37s$;
5) $\{a, (a+b)/2^{(2)}, (a+c)/2^{(4)}\}(2m)$,
$AC: \{(a+c)/2, (a+c)/2(a+b)/2\}\{m, 2m\}\{m(a+c)/2, 2m(a+c)/2(a+b)/2\}$;
6) $\{a, (a+b)/2^{(2)}, (a+c)/2^{(4)}\}(2m^{(2)})$,
$AC: \{(a+c)/2, (a+c)/2(a+b)/2\}\{m, 2m\}\{m(a+c)/2, 2m(a+c)/2(a+b)/2\}$;
7) $\{a, (a+b)/2(2), (a+c)/2(4)\}(2(2)m)$,
$AC: \{(a+c)/2\}\{(a+b)/2\}\{2\}\{m\}, XXIII, 37s$;
8) $\{a, (a+b)/2, (a+c)/2^{(3)}\}(2m^{(2)})$,
$AC: \{(a+c)/2, (a+c)/2(a+b)/2\}\{m, 2m\}\{m(a+c)/2, 2m(a+c)/2(a+b)/2\}$;
9) $\{a, (a+b)/2, (a+c)/2^{(3)}\}(2^{(2)}m)$,
$AC: \{(a+c)/2\}\{(a+b)/2\}\{2\}\{m\}, XXIII, 37s$;
10) $\{a, (a+b)/2, (a+c)/2^{(6)}\}(2m)$,
$AC: \{(a+c)/2, (a+c)/2(a+b)/2\}\{m, 2m\}\{m(a+c)/2, 2m(a+c)/2(a+b)/2\}$;
11) $\{a, (a+b)/2, (a+c)/2^{(6)}\}(2m^{(2)})$,
$AC: \{(a+c)/2, (a+c)/2(a+b)/2\}\{m, 2m\}\{m(a+c)/2, 2m(a+c)/2(a+b)/2\}$;
12) $\{a, (a+b)/2, (a+c)/2^{(6)}\}(2^{(2)}m)$,
$AC: \{(a+c)/2\}\{(a+b)/2\}\{2\}\{m\}, XXIII, 37s$;
13) $\{a, (a+b)/2^{(2)}, (a+c)/2^{(3)}\}(2m)$,
$AC: \{(a+c)/2\}\{(a+b)/2\}\{2\}\{m\}, XXIII, 37s$;
14) $\{a, (a+b)/2^{(2)}, (a+c)/2^{(3)}\}(2m^{(2)})$,
$AC: \{(a+c)/2\}\{(a+b)/2\}\{2\}\{m\}, XXIII, 37s$;
15) $\{a, (a+b)/2^{(2)}, (a+c)/2^{(3)}\}(2^{(2)}m)$,
$AC: \{(a+c)/2\}\{(a+b)/2\}\{2\}\{m\}, XXIII, 37s$;
16) $\{a, (a+b)/2^{(2)}, (a+c)/2^{(3)}\}(2^{(2)}m^{(2)})$,
$AC: \{(a+c)/2\}\{(a+b)/2\}\{2\}\{m\}, XXIII, 37s$.

$VI)$ $22s, P4, \{a, b, c\}(4)$, $AC: \{c\}\{4, 4a\}$;
1) $\{a, b, c(3)\}(4)$, $AC: \{c\}\{4, 4a\}$;
2) $\{a, b, c\}(4^{(4)})$, $AC: \{c\}\{4, 4a\}$;
3) $\{a, b, c^{(2)}\}(4^{(4)})$, $AC: \{c\}\{4, 4a\}$;
4) $\{a^{(2)}, b^{(2)}, c\}(4^{(4)})$, $AC: \{c\}\{4, 4a\}$;
5) $\{a^{(2)}, b^{(2)}, c^{(2)}\}(4^{(4)})$, $AC: \{c\}\{4, 4a\}$;
6) $\{a, b, c^{(4)}\}(4)$, $AC: \{c\}\{4, 4a\}$;
7) $\{a, b, c^{(4)}\}(4^{(2)})$, $AC: \{c\}\{4, 4a\}$;
8) $\{a, b, c^{(4)}\}(4^{(4)})$, $AC: \{c\}\{4, 4a\}$;
9) $\{a, b, c^{(4)}\}(4^{-(4)})$, $AC: \{c\}\{4, 4a\}$;
10) $\{a^{(2)}, b^{(2)}, c^{(4)}\}(4)$, $AC: \{a\}\{c\}\{4\}, XX, 25s$;
11) $\{a^{(2)}, b^{(2)}, c^{(4)}\}(4^{(4)})$, $AC: \{a\}\{c\}\{4\}, XX, 25s$;
12) $\{a, b, c^{(3)}\}(4^{(2)})$, $AC: \{c\}\{4, 4a\}$;
13) $\{a, b, c^{(6)}\}(4)$, $AC: \{c\}\{4, 4a\}$;
14) $\{a, b, c^{(6)}\}(4^{(2)})$, $AC: \{c\}\{4, 4a\}$;

15) $\{a^{(2)}, b^{(2)}, c^{(3)}\}(4)$, $AC : \{a\}\{c\}\{4\}, XX, 25s$;
16) $\{a^{(2)}, b^{(2)}, c^{(6)}\}(4)$, $AC : \{a\}\{c\}\{4\}, XX, 25s$.

$XIX)$ $23s, I4, \{a, b, (a+b+c)/2\}(4)$, $AC : \{(a+b+c)/2\}\{4\}$;
1) $\{a, b, (a+b+c)/2^{(3)}\}(4)$, $AC : \{(a+b+c)/2\}\{4\}$;
2) $\{a, b, (a+b+c)/2^{(4)}\}(4)$, $AC : \{(a+b+c)/2\}\{4\}$;
3) $\{a, b, (a+b+c)/2^{(2)}\}(4^{(4)})$, $AC : \{(a+b+c)/2\}\{4\}$;
4) $\{a, b, (a+b+c)/2^{(4)}\}(4)$, $AC : \{(a+b+c)/2\}\{4\}$;
5) $\{a, b, (a+b+c)/2^{(4)}\}(4^{(2)})$, $AC : \{(a+b+c)/2\}\{4\}$;
6) $\{a, b, (a+b+c)/2^{(4)}\}(4^{(4)})$, $AC : \{(a+b+c)/2\}\{4\}$;
7) $\{a, b, (a+b+c)/2^{(4)}\}(4^{-(4)})$, $AC : \{(a+b+c)/2\}\{4\}$;
8) $\{a, b, (a+b+c)/2^{(3)}\}(4^{(2)})$, $AC : \{(a+b+c)/2\}\{4\}$;
9) $\{a, b, (a+b+c)/2^{(6)}\}(4)$, $AC : \{(a+b+c)/2\}\{4\}$;
10) $\{a, b, (a+b+c)/2^{(6)}\}(4^{(2)})$, $AC : \{(a+b+c)/2\}\{4\}$.

$XIII)$ $24s, P4mm, \{a, b, c\}(4m)$, $AC : \{c\}\{m\}\{4, 4a\}$;
1) $\{a, b, c^{(3)}\}(4m)$, $AC : \{c\}\{m\}\{4, 4a\}$;
2) $\{a, b, c^{(4)}\}(4m)$, $AC : \{c\}\{m\}\{4, 4a\}$;
3) $\{a, b, c^{(4)}\}(4m^{(2)})$, $AC : \{c\}\{m\}\{4, 4a\}$;
4) $\{a, b, c^{(4)}\}(4^{(2)}m)$, $AC : \{c\}\{m\}\{4, 4a\}$;
5) $\{a, b, c^{(4)}\}(4^{(2)}m^{(2)})$, $AC : \{c\}\{m\}\{4, 4a\}$;
6) $\{a^{(2)}, b^{(2)}, c^{(4)}\}(4m)$, $AC : \{a\}\{c\}\{4\}\{m\}, XXIII, 37s$;
7) $\{a^{(2)}, b^{(2)}, c^{(4)}\}(4m^{(2)})$, $AC : \{a\}\{c\}\{4\}\{m\}, XXIII, 37s$;
8) $\{a, b, c^{(3)}\}(4m^{(2)})$, $AC : \{c\}\{m\}\{4, 4a\}$;
9) $\{a, b, c^{(3)}\}(4^{(2)}m)$, $AC : \{c\}\{m\}\{4, 4a\}$;
10) $\{a, b, c^{(3)}\}(4^{(2)}m^{(2)})$, $AC : \{c\}\{m\}\{4, 4a\}$;
11) $\{a, b, c^{(6)}\}(4m)$, $AC : \{c\}\{m\}\{4, 4a\}$;
12) $\{a, b, c^{(6)}\}(4m^{(2)})$, $AC : \{c\}\{m\}\{4, 4a\}$;
13) $\{a, b, c^{(6)}\}(4^{(2)}m)$, $AC : \{c\}\{m\}\{4, 4a\}$;
14) $\{a, b, c^{(6)}\}(4^{(2)}m^{(2)})$, $AC : \{c\}\{m\}\{4, 4a\}$;
15) $\{a^{(2)}, b^{(2)}, c^{(3)}\}(4m)$, $AC : \{a\}\{c\}\{4\}\{m\}, XXIII, 37s$;
16) $\{a^{(2)}, b^{(2)}, c^{(3)}\}(4m^{(2)})$, $AC : \{a\}\{c\}\{4\}\{m\}, XXIII, 37s$;
17) $\{a^{(2)}, b^{(2)}, c^{(6)}\}(4m)$, $AC : \{a\}\{c\}\{4\}\{m\}, XXIII, 37s$;
18) $\{a^{(2)}, b^{(2)}, c^{(6)}\}(4m^{(2)})$, $AC : \{a\}\{c\}\{4\}\{m\}, XXIII, 37s$;

$XX)$ $25s, I4mm, \{a, b, (a+b+c)/2\}(4m)$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;
1) $\{a, b, (a+b+c)/2^{(3)}\}(4m)$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;
2) $\{a, b, (a+b+c)/2^{(4)}\}(4m)$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;
3) $\{a, b, (a+b+c)/2^{(4)}\}(4m^{(2)})$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;
4) $\{a, b, (a+b+c)/2^{(4)}\}(4^{(2)}m)$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;
5) $\{a, b, (a+b+c)/2^{(4)}\}(4^{(2)}m^{(2)})$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;
6) $\{a, b, (a+b+c)/2^{(3)}\}(4m^{(2)})$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;
7) $\{a, b, (a+b+c)/2^{(3)}\}(4^{(2)}m)$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;
8) $\{a, b, (a+b+c)/2^{(3)}\}(4^{(2)}m^{(2)})$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;
9) $\{a, b, (a+b+c)/2^{(6)}\}(4m)$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;
10) $\{a, b, (a+b+c)/2^{(6)}\}(4m^{(2)})$, $AC : \{(a+b+c)/2\}\{4\}\{m\}$;

11) $\{a,b,(a+b+c)/2^{(6)}\}(4^{(2)}m)$, $AC:\{(a+b+c)/2\}\{4\}\{m\}$;
12) $\{a,b,(a+b+c)/2^{(6)}\}(4^{(2)}m^{(2)})$, $AC:\{(a+b+c)/2\}\{4\}\{m\}$;

$IV)$ $26s, P\bar{4}, \{a,b,c)(\tilde{4})$, $AC:\{\tilde{4},\tilde{4}b\}\{\tilde{4}c,\tilde{4}bc\}$;
1) $\{a,b,c)(\tilde{4}^{(4)})$, $AC:\{\tilde{4},\tilde{4}b\}\{\tilde{4}c,\tilde{4}bc\}$;
2) $\{a,b,c^{(2)})(\tilde{4}^{(4)})$, $AC:\{\tilde{4},\tilde{4}b\}\{\tilde{4}c,\tilde{4}bc\}$;
3) $\{a^{(2)},b^{(2)},c)(\tilde{4}^{(4)})$, $AC:\{c\}\{\tilde{4},\tilde{4}b\}, VI, 6s$;
4) $\{a^{(2)},b^{(2)},c^{(2)})(\tilde{4}^{(4)})$, $AC:\{c\}\{\tilde{4},\tilde{4}b\}, VI, 6s$.

$XXI)$ $27s, I\bar{4}, \{a,b,(a+b+c)/2\}(\tilde{4})$, $AC:\{\tilde{4},\tilde{4}(a+b+c)/2\}$;
1) $\{a,b,(a+b+c)/2\}(\tilde{4}^{(4)})$, $AC:\{\tilde{4},\tilde{4}(a+b+c)/2\}$;
2) $\{a,b,(a+b+c)/2^{(2)}\}(\tilde{4}^{(4)})$, $AC:\{\tilde{4},\tilde{4}(a+b+c)/2\}$;
3) $\{a^{(2)},b^{(2)},(a+b+c)/2^{(4)}\}(\tilde{4})$, $AC:\{\tilde{4}\}\{(a+b+c)/2\}, XIX, 23s$.

$XXII)$ $28s, P4/m, \{a,b,c\}(4:m)$, $AC:\{4,4a\}\{m,cm\}$;
1) $\{a,b,c\}(4^{(4)}:m)$, $AC:\{4,4a\}\{m,cm\}$;
2) $\{a,b,c\}(4^{(4)}:m^{(2)})$, $AC:\{4,4a\}\{m,cm\}$;
3) $\{a,b,c^{(2)}\}(4^{(4)}:m)$, $AC:\{4,4a\}\{m,cm\}$;
4) $\{a^{(2)},b^{(2)},c\}(4^{(4)}:m)$, $AC:\{4,4a\}\{m,cm\}$;
5) $\{a^{(2)},b^{(2)},c\}(4^{(4)}:m^{(2)})$, $AC:\{4,4a\}\{m,cm\}$;
6) $\{a^{(2)},b^{(2)},c^{(2)}\}(4^{(4)}:m)$, $AC:\{c\}\{m\}\{4,4a\}, XIII, 14s$.

$XX)$ $29s, I4/m, \{a,b,(a+b+c)/2\}(4:m)$, $AC:\{(a+b+c)/2\}\{4\}\{2\}$;
1) $\{a,b,(a+b+c)/2\}(4^{(4)}:m)$, $AC:\{(a+b+c)/2\}\{4\}\{m\}$;
2) $\{a,b,(a+b+c)/2\}(4^{(4)}:m^{(2)})$, $AC:\{(a+b+c)/2\}\{4\}\{m\}$;
3) $\{a,b,(a+b+c)/2^{(2)}\}(4^{(4)}:m)$, $AC:\{(a+b+c)/2\}\{4\}\{m\}$;
4) $\{a,b,(a+b+c)/2^{(2)}\}(4^{(4)}:m^{(2)})$, $AC:\{(a+b+c)/2\}\{4\}\{m\}$.

$XX)$ $34s, I\bar{4}2m, \{a,b,(a+b+c)/2\}(\tilde{4}:2)$, $AC:\{(a+b+c)/2\}\{\tilde{4}\}\{2\}$;
1) $\{a^{(2)},b^{(2)},(a+b+c)/2^{(4)}\}(\tilde{4}:2)$, $AC:\{(a+b+c)/2\}\{\tilde{4}\}\{2\}$;
2) $\{a^{(2)},b^{(2)},(a+b+c)/2^{(4)}\}(\tilde{4}:2^{(2)})$, $AC:\{(a+b+c)/2\}\{\tilde{4}\}\{2\}$.

$XXIV)$ $38s, P3, \{(a,b),c\}^{(3)}$, $AC:\{c\}$;
1) $\{(a,b),c\}(3^{(3)})$, $AC:\{c\}$;
2) $\{(a,b),c^{(3)}\}^{(3)}$, $AC:\{c\}$;
3) $\{(a,b),c^{(3)}\}(3^{(3)})$, $AC:\{c\}$;
4) $\{(a,b),c^{(3)}\}(3(-3))$, $AC:\{c\}$;
5) $\{(a^{(3)},b^{(3)}),c\}^{(3)}$, $AC:\{c\}$;
6) $\{(a^{(3)},b^{(3)}),c^{(3)}\}^{(3)}$, $AC:\{c\}$;
7) $\{(a,b),c^{(4)}\}^{(3)}$, $AC:\{c\}$;
8) $\{(a,b),c^{(2)}\}(3^{(3)})$, $AC:\{c\}$;
9) $\{(a,b),c^{(6)}\}^{(3)}$, $AC:\{c\}$;
10) $\{(a,b),c^{(6)}\}(3^{(3)})$, $AC:\{c\}$;
11) $\{(a,b),c^{(6)}\}(3(-3))$, $AC:\{c\}$;
12) $\{(a^{(3)},b^{(3)}),c^{(2)}\}^{(3)}$, $AC:\{c\}$;
13) $\{(a^{(3)},b^{(3)}),c^{(6)}\}^{(3)}$, $AC:\{c\}$.

$XXIV)$ $39s, R3\{a,b,c\}^{(3)}$, $AC:\{a\}$;
1) $\{a,b,c\}(3^{(3)})$, $AC:\{a\}$;
2) $\{a^{(3)},b^{(3)},c^{(3)}\}^{(3)}$, $AC:\{a\}$;

3) $\{a^{(3)}, b^{(3)}, c^{(3)}\}(3^{(3)})$, $AC : \{a\}$;
4) $\{a^{(3)}, b^{(3)}, c^{(3)}\}(3(-3))$, $AC : \{a\}$;
5) $\{a^{(4)}, b^{(4)}, c^{(4)}\}^{(3)}$, $AC : \{a\}$;
6) $\{a^{(2)}, b^{(2)}, c^{(2)}\}(3^{(3)})$, $AC : \{a\}$;
7) $\{a^{(6)}, b^{(6)}, c^{(6)}\}^{(3)}$, $AC : \{a\}$;
8) $\{a^{(6)}, b^{(6)}, c^{(6)}\}(3^{(3)})$, $AC : \{a\}$;
9) $\{a^{(6)}, b^{(6)}, c^{(6)}\}(3(-3))$, $AC : \{a\}$.

XIX) $40s, P3m1, \{(a,b),c\}(3m)$, $AC : \{c\}\{m\}$;
1) $\{(a,b), c^{(3)}\}(3m)$, $AC : \{c\}\{m\}$;
2) $\{(a,b), c^{(4)}\}(3m)$, $AC : \{c\}\{m\}$;
3) $\{(a,b), c^{(4)}\}(3m^{(2)})$, $AC : \{c\}\{m\}$;
4) $\{(a,b), c^{(3)}\}(3m^{(2)})$, $AC : \{c\}\{m\}$;
5) $\{(a,b), c^{(6)}\}(3m)$, $AC : \{c\}\{m\}$;
6) $\{(a,b), c^{(6)}\}(3m^{(2)})$, $AC : \{c\}\{m\}$.

XIX) $41s, P31m, \{(a,b),c\}(m3)$, $AC : \{c\}\{m\}$;
1) $\{(a,b), c^{(3)}\}(m3)$, $AC : \{c\}\{m\}$;
2) $\{(a^{(3)}, b^{(3)}), c\}(m3)$, $AC : \{c\}\{m\}$;
3) $\{(a^{(3)}, b^{(3)}), c^{(3)}\}(m3)$, $AC : \{c\}\{m\}$;
4) $\{(a,b), c^{(4)}\}(m3)$, $AC : \{c\}\{m\}$;
5) $\{(a,b), c^{(4)}\}(m^{(2)}3)$, $AC : \{c\}\{m\}$;
6) $\{(a,b), c^{(3)}\}(m^{(2)}3)$, $AC : \{c\}\{m\}$;
7) $\{(a^{(3)}, b^{(3)}), c\}(m^{(2)}3)$, $AC : \{c\}\{m\}$;
8) $\{(a^{(3)}, b^{(3)}), c^{(3)}\}(m^{(2)}3)$, $AC : \{c\}\{m\}$;
9) $\{(a,b), c^{(6)}\}(m3)$, $AC : \{c\}\{m\}$;
10) $\{(a,b), c^{(6)}\}(m^{(2)}3)$, $AC : \{c\}\{m\}$;
11) $\{(a^{(3)}, b^{(3)}), c^{(2)}\}(m3)$, $AC : \{c\}\{m\}$;
12) $\{(a^{(3)}, b^{(3)}), c^{(2)}\}(m^{(2)}3)$, $AC : \{c\}\{m\}$;
13) $\{(a^{(3)}, b^{(3)}), c^{(6)}\}(m3)$, $AC : \{c\}\{m\}$;
14) $\{(a^{(3)}, b^{(3)}), c^{(6)}\}(m^{(2)}3)$, $AC : \{c\}\{m\}$.

XIX) $42s, R3m\{a,b,c\}(m3)$, $AC : \{a\}\{m\}$;
1) $\{a^{(3)}, b^{(3)}, c^{(3)}\}(m3)$, $AC : \{a\}\{m\}$;
2) $\{a^{(4)}, b^{(4)}, c^{(4)}\}(m3)$, $AC : \{a\}\{m\}$;
3) $\{a^{(4)}, b^{(4)}, c^{(4)}\}(m^{(2)}3)$, $AC : \{a\}\{m\}$;
4) $\{a^{(3)}, b^{(3)}, c^{(3)}\}(m^{(2)}3)$, $AC : \{a\}\{m\}$;
5) $\{a^{(6)}, b^{(6)}, c^{(6)}\}(m3)$, $AC : \{a\}\{m\}$;
6) $\{a^{(6)}, b^{(6)}, c^{(6)}\}(m^{(2)}3)$, $AC : \{a\}\{m\}$.

XXI) $43s, P6, \{(a,b),c\}(3:m)$, $AC : \{m, cm\}$;
1) $\{(a,b),c\}(3^{(3)}:m)$, $AC : \{m, cm\}$;
2) $\{(a^{(3)}, b^{(3)}), c\}(3:m)$, $AC : \{m, cm\}$;
3) $\{(a,b),c\}(3^{(3)}:m^{(2)})$, $AC : \{m, cm\}$;
4) $\{(a,b), c^{(2)}\}(3^{(3)}:m)$, $AC : \{c\}\{m\}, XIX, 23s$;
5) $\{(a^{(3)}, b^{(3)}), c\}(3:m^{(2)})$, $AC : \{m, cm\}$;
6) $\{(a^{(3)}, b^{(3)}), c^{(2)}\}(3:m)$, $AC : \{c\}\{m\}, XIX, 23s$.

$XXI)$ $44s, P321, \{(a,b),c\}(3:2)$, $AC:\{2,2c\}$;
1) $\{(a^{(3)},b^{(3)}),c\}(3:2)$, $AC:\{2,2c\}$;
2) $\{(a^{(3)},b^{(3)}),c\}(3:2^{(2)})$, $AC:\{2,2c\}$;
3) $\{(a^{(3)},b^{(3)}),c^{(2)}\}(3:2)$, $AC:\{c\}\{2\}, XIX, 23s$.

$VI)$ $47s, P62m, \{(a,b),c\}(3:m2)$, $AC:\{m)\{2,2c\}$;
1) $\{(a^{(3)},b^{(3)}),c\}(3:m2)$, $AC:\{m\}\{2,2c\}$;
2) $\{(a^{(3)},b^{(3)}),c\}(3:m^{(2)}2)$, $AC:\{m\}\{2,2c\}$;
3) $\{(a^{(3)},b^{(3)}),c\}(3:m2^{(2)})$, $AC:\{m\}\{2,2c\}$;
4) $\{(a^{(3)},b^{(3)}),c\}(3:m^{(2)}2^{(2)})$, $AC:\{m\}\{2,2c\}$;
5) $\{(a^{(3)},b^{(3)}),c^{(2)}\}(3:m2)$, $AC:\{c\}\{m\}\{2\}, XX, 25s$;
6) $\{(a^{(3)},b^{(3)}),c^{(2)}\}(3:m^{(2)}2)$, $AC:\{c\}\{m\}\{2\}, XX, 25s$.

$XIX)$ $49s, P6, \{(a,b),c\}(6)$, $AC:\{c\}\{6\}$;
1) $\{(a,b),c\}(6^{(3)})$, $AC:\{c\}\{6\}$;
2) $\{(a,b),c^{(3)}\}(6)$, $AC:\{c\}\{6\}$;
3) $\{(a,b),c^{(3)}\}(6^{(3)})$, $AC:\{c\}\{6\}$;
4) $\{(a,b),c^{(3)}\}(6(-3))$, $AC:\{c\}\{6\}$;
5) $\{(a,b),c^{(4)}\}(6)$, $AC:\{c\}\{6\}$;
6) $\{(a,b),c^{(4)}\}(6^{(2)})$, $AC:\{c\}\{6\}$;
7) $\{(a,b),c\}(6^{(6)})$, $AC:\{c\}\{6\}$;
8) $\{(a,b),c^{(2)}\}(6^{(3)})$, $AC:\{c\}\{6\}$;
9) $\{(a,b),c^{(2)}\}(6^{(6)})$, $AC:\{c\}\{6\}$;
10) $\{(a,b),c^{(3)}\}(6^{(2)})$, $AC:\{c\}\{6\}$;
11) $\{(a,b),c^{(3)}\}(6^{(6)})$, $AC:\{c\}\{6\}$;
12) $\{(a,b),c^{(3)}\}(6(-6))$, $AC:\{c\}\{6\}$;
13) $\{(a,b,c^{(6)}\}(6)$, $AC:\{c\}\{6\}$;
14) $\{(a,b),c^{(6)}\}(6^{(2)})$, $AC:\{c\}\{6\}$;
15) $\{(a,b),c^{(6)}\}(6^{(3)}$ $AC:\{c\}\{6\}$;
16) $\{(a,b),c^{(6)}\}(6(-3))$, $AC:\{c\}\{6\}$;
17) $\{(a,b),c^{(6)}\}(6^{(6)})$, $AC:\{c\}\{6\}$;
18) $\{(a,b,c^{(6)}\}(6(-6))$, $AC:\{c\}\{6\}$.

$XX)$ $50s, P6mm, \{(a,b),c\}(6m)$, $AC:\{c\}\{6\}\{m\}$;
1) $\{(a,b),c^{(3)}\}(6m)$, $AC:\{c\}\{6\}\{m\}$;
2) $\{(a,b),c^{(4)}\}(6m)$, $AC:\{c\}\{6\}\{m\}$;
3) $\{(a,b),c^{(4)}\}(6m^{(2)})$, $AC:\{c\}6\{m\}$;
4) $\{(a,b),c^{(4)}\}(6^{(2)}m)$, $AC:\{c\}\{6\}\{m\}$;
5) $\{(a,b),c^{(4)}\}(6^{(2)}m^{(2)})$, $AC:\{c\}\{6\}\{m\}$;
6) $\{(a,b),c^{(3)}\}(6m^{(2)})$, $AC:\{c\}\{6\}\{m\}$;
7) $\{(a,b),c^{(3)}\}(6^{(2)}m)$, $AC:\{c\}\{6\}\{m\}$;
8) $\{(a,b),c^{(3)}\}(6^{(2)}m^{(2)})$, $AC:\{c\}\{6\}\{m\}$;
9) $\{(a,b),c^{(6)}\}(6m)$, $AC:\{c\}\{6\}\{m\}$;
10) $\{(a,b),c^{(6)}\}(6m^{(2)})$, $AC:\{c\}\{6\}\{m\}$;
11) $\{(a,b),c^{(6)}\}(6^{(2)}m)$, $AC:\{c\}\{6\}\{m\}$;
12) $\{(a,b),c^{(6)}\}(6^{(2)}m^{(2)})$, $AC:\{c\}\{6\}\{m\}$.

$XXI)$	$51s, P3, \{(a,b),c\}(6)$,	$AC : \{6, 6c\}$;
1)	$\{(a,b),c\}(6^{(3)})$,	$AC : \{6, 6c\}$;
2)	$\{(a,b),c\}(6^{(6)})$,	$AC : \{6, 6c\}$;
3)	$\{(a,b,c^{(2)}\}(6^{(3)})$,	$AC : \{c\}\{6\}, XIX, 23s$.
$XXI)$	$52s, R3\{a,b,c\}(6)$,	$AC : \{6, 6a\}$;
1)	$\{a,b,c\}(6^{(3)})$,	$AC : \{6, 6a\}$;
2)	$\{a,b,c\}(6^{(6)})$,	$AC : \{6, 6a\}$;
3)	$\{a^{(2)}, b^{(2)}, c^{(2)}\}(6^{(3)})$,	$AC : \{a\}\{6\}, XIX, 23s$.
$VI)$	$53s, P6/m, \{(a,b),c\}(6 : m)$,	$AC : \{6\}\{m, cm\}$;
1)	$\{(a,b),c\}(6^{(3)} : m)$,	$AC : \{6\}\{m, cm\}$;
2)	$\{(a,b),c\}(6^{(3)} : m^{(2)})$,	$AC : \{6\}\{m, cm\}$;
3)	$\{(a,b),c^{(2)}\}(6^{(3)} : m)$,	$AC : \{c\}\{6\}\{m\}, XX, 25s$;
4)	$\{(a,b),c^{(2)}\}(6^{(6)} : m)$,	$AC : \{c\}\{6\}\{m\}, XX, 25s$;
5)	$\{(a,b),c\}(6^{(6)} : m)$,	$AC : \{6\}\{m, cm\}$;
6)	$\{(a,b),c\}(6^{(2)} : m^{(2)})$,	$AC : \{6\}\{m, cm\}$.
$XXIV)$	$59s, P23, \{a,b,c\}(3/2)$,	$AC : \{a\}$;
1)	$\{a,b,c\}(3^{(3)}/2)$,	$AC : \{a\}$;
2)	$\{a^{(2)}, b^{(2)}, c^{(2)}\}(3^{(3)}/2)$,	$AC : \{a\}$.
$XXIV)$	$60s, I23, \{a,b,(a+b+c)/2\}(3/2)$,	$AC : \{(a+b+c)/2\}$;
1)	$\{a,b,(a+b+c)/2\}(3^{(3)}/2)$,	$AC : \{(a+b+c)/2\}$;
2)	$\{a^{(2)}, b^{(2)}, (a+b+c)/2^{(4)}\}(3/2)$,	$AC : \{(a+b+c)/2\}$;
3)	$\{a,b,(a+b+c)/2^{(2)}\}(3^{(3)}/2)$,	$AC : \{(a+b+c)/2\}$.
$XXV)$	$61s, F23, \{a,(a+b)/2,(a+c)/2\}(3/2)$;	
1)	$\{a,(a+b)/2,(a+c)/2\}(3^{(3)}/2)$.	
$XXI)$	$62s, Pm3, \{a,b,c\}(3/2m)$,	$AC : \{m, ma\}$;
1)	$\{a,b,c\}(3^{(3)}/2m)$,	$AC : \{m, ma\}$;
2)	$\{a,b,c\}(3^{(3)}/2m^{(2)})$,	$AC : \{m, ma\}$;
3)	$\{a^{(2)}, b^{(2)}, c^{(2)}\}(3^{(3)}/2m)$,	$AC : \{a\}\{m\}, XIX, 23s$.
$XIX)$	$63s, Im3, \{a,b,(a+b+c)/2\}(3/2m)$,	$AC : \{(a+b+c)/2\}\{m\}$;
1)	$\{a,b,(a+b+c)/2\}(3^{(3)}/2m)$,	$AC : \{(a+b+c)/2\}\{m\}$;
2)	$\{a,b,(a+b+c)/2\}(3^{(3)}/2m^{(2)})$,	$AC : \{(a+b+c)/2\}\{m\}$;
3)	$\{a,b,(a+b+c)/2^{(2)}\}(3^{(3)}/2m)$,	$AC : \{(a+b+c)/2\}\{m\}$;
4)	$\{a,b,(a+b+c)/2^{(2)}\}(3^{(3)}/2m^{(2)})$,	$AC : \{(a+b+c)/2\}$.
$XXIV)$	$64s, Fm3, \{a,(a+b)/2,(a+c)/2\}(3/2m)$,	$AC : \{m\}$;
1)	$\{a,(a+b)/2,(a+c)/2\}(3^{(3)}/2m)$,	$AC : \{m\}$;
2)	$\{a,(a+b)/2,(a+c)/2\}(3^{(3)}/2m^{(2)})$,	$AC : \{m\}$.
$XIX)$	$66s, I43m, \{a,b,(a+b+c)/2\}(3/4)$,	$AC : \{(a+b+c)/2\}\{4\}$;
1)	$\{a^{(2)}, b^{(2)}, (a+b+c)/2^{(4)}\}(3/4)$,	$AC : \{(a+b+c)/2\}\{4\}$;
2)	$\{a^{(2)}, b^{(2)}, (a+b+c)/2^{(4)}\}(3/4^{(2)})$,	$AC : \{(a+b+c)/2\}\{4\}$.

From Table 1 is clear that the almost all AC of junior symmorphic p-symmetry space groups are isomorphic to the already investigated AC [10,11] given in the partial catalogue of AC of the classical-symmetry Fedorov groups G_3 [11, Appendix]. The only exceptions are a few junior p-symmetry groups from the family $13s$, which are investigated independently.

Extending the proposition that groups possessing isomorphic AC generate the same number of (2^l)-symmetry groups of the M^m-type which are corresponding in the sense of structure [10,11,12] on p-generating groups, we have the numbers $N_m(G)$ given in Table 2, where by G is denoted the representative of the corresponding AC-isomorphism equivalence class.

Table 2

G	$N_1(G)$	$N_2(G)$	$N_3(G)$	$N_4(G)$	$N_5(G)$
I	1	1	1		
III	5	28	168	840	
IV	4	15	42		
V	5	34	266	1680	
VI	5	24	84		
$VIII$	9	84	756	5040	
X	3	10	28		
XII	11	186	3948	83160	1249920
$XIII$	11	126	1344	10080	
XIV	9	108	1260	10080	
XVI	17	348	7812	166320	2499840
$XVII$	7	58	504	3360	
XIX	3	6			
XX	7	42	168		
XXI	2	3			
$XXII$	8	75	714	5040	
$XXIII$	15	210	2520	201	
$XXIV$	1				
$XXXI$	2	4	7		
13s.5.	23	570	14280	322560	4999680
13s.7.	19	486	13104	312480	4999680

For $p = 3$ holds the relationship $N_m(G) = (N_m(G))$. For $p = 4, 6$, the corresponding numbers $(N_m(G))$ $(p = 4, 6)$ are given in Table 3 (see also [11]).

Table 3

G	$(N_1(G))$	$(N_2(G))$	$(N_3(G))$	$(N_4(G))$
III	4	16	56	
IV	3	6		
V	4	22	112	
VI	4	12		
$VIII$	8	60	336	
X	2	4		
XII	10	156	2856	40320
$XIII$	10	96	672	
XIV	8	84	672	
XVI	16	300	5712	80640
$XVII$	6	40	224	
XIX	2			
XX	6	24		
XXI	1			
$XXII$	7	54	336	
$XXIII$	14	168	1344	
$XXXI$	1	1		
13s.5.	22	504	10752	161280
13s.7.	18	432	10080	161280

In Table 4 is given the distribution of 316 p-generating symmorphic space groups according to the AC-isomorphism equivalence classes.

Table 4

G	$p=3$	$p=4$	$p=6$	$p=4,6$	$p=3,4,6$
I	1				1
III	1	2	3	5	6
IV	1	4	3	7	8
V	1		1	1	2
VI	5	15	15	30	35
$VIII$		1	2	3	3
X		1		1	1
XII	1	2	3	5	6
$XIII$	3	15	22	37	40
XIV	1	4	3	7	8
XVI		1	2	3	3
$XVII$			1	1	1
XIX	11	17	39	56	67
XX	2	19	25	44	46
XXI	6	2	6	8	14
$XXII$		6	2	8	8
$XXIII$		8	18	26	26
$XXIV$	13	3	13	16	29
XXV	1				1
$XXXI$		1	1	2	2
$13s.5.$		2	4	6	6
$13s.7.$		1	2	3	3
	47	104	165	269	316

Multiplying the number of the groups belonging to a certain AC-isomorphism equivalence class by the corresponding number $N_m(G)$ or $(N_m(G))$, and adding the products obtained, we have the numbers N_m and (N_m) of the symmorphic space groups of $(p, 2^l)$-symetry ($p = 3, 4, 6$).

$$N_1 = 165^{(3)} + 776^{(4)} + 1203^{(6)} = 2144$$
$$(N_1) = 165^{(3)} + 672^{(4)} + 1038^{(6)} = 1875$$
$$N_2 = 1038^{(3)} + 8227^{(4)} + 13587^{(6)} = 22852$$
$$(N_2) = 1038^{(3)} + 6211^{(4)} + 10473^{(6)} = 17764$$
$$N_3 = 10473^{(3)} + 112133^{(4)} + 199311^{(6)} = 321917$$
$$(N_3) = 10473^{(3)} + 68656^{(4)} + 126000^{(6)} = 205129$$
$$N_4 = 126000^{(3)} + 1674960^{(4)} + 3139920^{(6)} = 4940880$$
$$(N_4) = 126000^{(3)} + 645120^{(4)} + 1249920^{(6)} = 2021040$$
$$N_5 = 1249920^{(3)} + 19998720^{(4)} + 38747520^{(6)} = 59996160$$
$$(N_5) = 1249920^{(3)} = 1249920$$

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