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On a Class of Operators with Normal Aluthge Transformations

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Abstract. In this paper, we show that the generalized Aluthge transformations of a large class of operators (weighted conditional type operators) are normal. As a consequence, the operator M_wEM_u is p-hyponormal if and only if it is normal, and under a weak condition, if M_wEM_u is normal, then the Holder inequality turn into equality for w, u. Also, we give some applications of p-hyponormal weighted conditional type operators, for instance, point spectrum and joint point spectrum of p-hyponormal weighted conditional type operators are equal. In the end, some examples are provided to illustrate concrete application of the main results of the paper.

1. Introduction

As is well-known, conditional expectation operators on various function spaces exhibit a number of remarkable properties related either to the underlying order structure, or to the metric structure when the function space is equipped with a norm. Such operators are necessarily positive projections which are averaging in a precise sense to be described below and in certain normed function space are contractive for the given norm. Theory of multiplication conditional type operators is one of important arguments in the connection of operator theory and measure theory. Conditional expectations have been studied in an operator theoretic setting, by, for example, De pagter and Grobler [6] and Rao [11, 12], as positive operators acting on L^p-spaces or Banach function spaces. In [10], S.-T. C. Moy characterized all operators on L^p of the form $f \to E(fg)$ for g in L^q with $E(|g|)$ bounded. Also, some results about multiplication conditional expectation operators can be found in [1, 7, 9]. In [2] P.G. Dodds, C.B. Huijsmans and B. de Pagter showed that lots of operators are of the form of weighted conditional type operators. The class of weighted conditional type operators contains composition operators, multiplication operators, weighted composition operators, some integral type operators and etc. These are some reasons that stimulate us to consider weighted conditional type operators in our work. So, in [3–5] we investigated some classic properties of multiplication conditional expectation operators M_wEM_u on L^p spaces. In this paper we state that the Aluthge transformations of weighted conditional type operators are normal, so we use this property and give some applications.

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2. Preliminaries

Let (X, Σ, μ) be a complete σ-finite measure space. For any sub-σ-finite algebra $\mathcal{A} \subseteq \Sigma$, the L^2-space $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated by $L^2(\mathcal{A})$, and its norm is denoted by $\|.\|_2$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function f is defined as $S(f) = \{x \in X; f(x) \neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on X by $L^0(\Sigma)$.

For a sub-σ-finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}} f$, defined for all non-negative measurable functions f as well as for all $f \in L^2(\Sigma)$, where $E^{\mathcal{A}} f$, by the Radon-Nikodym theorem, is the unique $\mathcal{A}$-measurable function satisfying $\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}$. As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^2(\Sigma)) = L^2(\mathcal{A})$. This operator will play a major role in our work. Let $f \in L^0(\Sigma)$, then f is said to be conditionable with respect to E if $f \in \mathcal{D}(E) := \{g \in L^0(\Sigma) : E(|g|) \in L^0(\mathcal{A})\}$. Throughout this paper we take u and w in $\mathcal{D}(E)$. If there is no possibility of confusion, we write $E(f)$ in place of $E^{\mathcal{A}}(f)$. A detailed discussion about this operator may be found in [13].

Let $\mathcal{H}$ be the infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. It is known that an operator T on a Hilbert space is p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, for $0 < p < \infty$. Every operator T on a Hilbert space $\mathcal{H}$ can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = (T^*T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(|T|)$. Then this decomposition is called the polar decomposition. The Aluthge transformation $\widehat{T}$ of the operator T is defined by $\widehat{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$. The operator T is said to be positive operator and write $T \geq 0$, if $\langle Th, h\rangle \geq 0$, for all $h \in \mathcal{H}$.
In this paper we will be concerned with normality, p-hyponormality and invertibility of weighted conditional type operators.

3. Main Results

In the first we reminisce some properties of weighted conditional type operators, that we have proved in [4].

The operator $T = M_w E M_u$ is bounded on $L^2(\Sigma)$ if and only if $(E|w|^2)^{\frac{1}{2}}(E|u|^2)^{\frac{1}{2}} \in L^\infty(\mathcal{A})$, and in this case its norm is given by $\|T\| = \|(E(|w|^2))^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_\infty$. The unique polar decomposition of bounded operator $T = M_w E M_u$ is $U|T|$, where

$$|T|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \chi_S \bar{u} E(uf)$$

and

$$U(f) = \left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}} w E(uf),$$

for all $f \in L^2(\Sigma)$, where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$. Also, the Aluthge transformation of $T = M_w E M_u$ is

$$\widehat{T}(f) = \frac{\chi_S E(uw)}{E(|u|^2)} \bar{u} E(uf), \qquad f \in L^2(\Sigma).$$

For each $\epsilon > 0$ we have

$$\widehat{T_\epsilon}(f) = |T|^\epsilon U |T|^{1-\epsilon}(f) = (E(|u|^2))^{-1} \chi_S E(uw) \overline{u} E(uf) = \widehat{T},$$

$$\widehat{T}^*(f) = (E(|u|^2))^{-1}\chi_S u\overline{E(uw)}E(uf),$$

and

$$|\widehat{T}|(f) = |\widehat{T}^*|(f) = E(|u|^2))^{-1}\chi_S|E(uw)|\bar{u}E(uf).$$

By above information we have a nice conclusion about conditional type operators $T = M_wEM_u$. For every conditional type operator $T = M_wEM_u$, the Aluthge transformation $\widehat{T}$ is normal.

Now we consider generalized Aluthge transformation of $T = M_wEM_u$. Let $T = U|T|$ be the polar decomposition of $T = M_wEM_u$. For $r > 0$ and $r \geq t \geq 0$, let $\widetilde{T} = |T|^tU|T|^{r-t}$. By [Lemma 3.3, [4]], for all $f \in L^2(\Sigma)$ we have

$$\widetilde{T}(f) = E(|w|^2)^{\frac{r-1}{2}}E(|u|^2)^{\frac{r-3}{2}}\chi_{S\cap G}E(uw)\bar{u}E(uf).$$

Theorem 3.1. For every multiplication conditional type operator $T = M_wEM_u$, the operator $\widetilde{T}$ is normal.

Proof Direct computations show that

$$\widetilde{T}^*\widetilde{T}(f) = E(|w|^2)^{r-1}E(|u|^2)^{r-2}\chi_{S\cap G}|E(uw)|^2\bar{u}E(uf) = \widetilde{T}\widetilde{T}^*(f).$$

This means that $\widetilde{T}$ is normal.

Theorem 3.2. Let $T = M_wEM_u$ be multiplication conditional type operator. Then T is p-hyponormal if and only if T is normal.

Proof By Theorem 3.1 and [Theorem 3, [8]] we conclude that if T is p-hyponormal, then T is normal. The converse is clear.

Lemma 3.3. Let $T = M_wEM_u$. Then $|\widetilde{T}| = |\widetilde{T}^*|$.

Proof Since $\widetilde{T}$ and $\widetilde{T}^*$ are also weighted conditional type operators, then we get that for all $f \in L^2(\Sigma)$

$$|\widetilde{T}|(f) = E(|w|^2)^{\frac{r-1}{2}}E(|u|^2)^{\frac{r-3}{2}}\chi_{S\cap G}|E(uw)|\bar{u}E(uf) = |\widetilde{T}^*|(f).$$

Theorem 3.4. Let $T = M_wEM_u$ be p-hyponormal. Then

$$|\widetilde{T}| = |\widetilde{T}^*| = |T|^r.$$

Proof By using Lemma 3.3 and [Theorem 2, [8]] we have $|\widetilde{T}| = |\widetilde{T}^*| = |T|^r$.

Theorem 3.5. If $T = M_wEM_u$ is p-hyponormal, then $|E(uw)|^2 = E(|u|^2)E(|w|^2)$ on $S(E(u))$.

Proof Let $f \in L^2(\Sigma)$, by [Lemma 3.3, [4]] and Theorem 3.4, for all $f \in L^2(\Sigma)$ we have

$$(E(|u|^2))^{\frac{r-2}{2}}(E(|w|^2))^{\frac{r}{2}}\chi_{S\cap G}\bar{u}E(uf) = (E(|u|^2))^{\frac{r-3}{2}}(E(|w|^2))^{\frac{r-1}{2}}\chi_{S\cap G}|E(uw)|\bar{u}E(uf)$$

and so for $0 < a \in L^2(\mathcal{A})$

$$((E(|u|^2))^{\frac{r-2}{2}}(E(|w|^2))^{\frac{r}{2}} - (E(|u|^2))^{\frac{r-3}{2}}(E(|w|^2))^{\frac{r-1}{2}}|E(uw)|)\chi_{S\cap G}\bar{u}E(u)a = 0.$$

By taking E we have

$$((E(|u|^2))^{\frac{1}{2}}(E(|w|^2))^{\frac{1}{2}} - |E(uw)|)(E(|u|^2))^{\frac{r-3}{2}}(E(|w|^2))^{\frac{r-1}{2}}\chi_{S\cap G}|E(u)|^2 a = 0.$$

And then

$$((E(|u|^2))^{\frac{1}{2}}(E(|w|^2))^{\frac{1}{2}} - |E(uw)|)|E(u)|^2 = 0.$$

This implies that $|E(uw)|^2 = E(|u|^2)E(|w|^2)$ on $S(E(u))$.

Corollary 3.6. If $T = M_wEM_u$ is p-hyponormal and $S(E(u)) = X$, then the conditional type Holder inequality for u, w turns into equality, i.e, $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.

From now on, we shall denote by $\sigma_p(T)$, $\sigma_{jp}(T)$, the point spectrum of T, the joint point spectrum of T, respectively. A complex number $\lambda \in \mathbb{C}$ is said to be in the point spectrum $\sigma_p(T)$ of the operator T, if there is a unit vector x satisfying $(T - \lambda)x = 0$. If in addition, $(T^* - \bar{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. For $A, B \in \mathcal{B}(\mathcal{H})$, it is well known that

$$\sigma_p(AB) \setminus \{0\} = \sigma_p(BA) \setminus \{0\}, \quad \sigma_{jp}(AB) \setminus \{0\} = \sigma_{jp}(BA) \setminus \{0\}.$$

Let $A_\lambda = \{x \in X : E(u)(x) = \lambda\}$, for $0 \neq \lambda \in \mathbb{C}$. Suppose that $\mu(A_\lambda) > 0$. Since $\mathcal{A}$ is σ-finite, there exists an $\mathcal{A}$-measurable subset B of A_λ such that $0 < \mu(B) < \infty$, and $f = \chi_B \in L^p(\mathcal{A}) \subseteq L^p(\Sigma)$. Now

$$EM_u(f) - \lambda f = E(u)\chi_B - \lambda\chi_B = 0.$$

This implies that $\lambda \in \sigma_p(EM_u)$.

If there exists $f \in L^p(\Sigma)$ such that $f\chi_C \neq 0$ μ-a.e, for $C \in \Sigma$ of positive measure and $E(uf) = \lambda f$ for $0 \neq \lambda \in \mathbb{C}$, then $f = \frac{E(uf)}{\lambda}$, which means that f is $\mathcal{A}$-measurable. Therefore $E(uf) = E(u)f = \lambda f$ and $(E(u) - \lambda)f = 0$. This implies that $C \subseteq A_\lambda$ and so $\mu(A_\lambda) > 0$. Hence

$$\sigma_p(EM_u) = \{\lambda \in \mathbb{C} : \mu(A_\lambda) > 0\}.$$

Thus

$$\sigma_p(M_wEM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} : \mu(A_{\lambda,w}) > 0\} \setminus \{0\},$$

where $A_{\lambda,w} = \{x \in X : E(uw)(x) = \lambda\}$.

Theorem 3.7. If $|E(uw)|^2 = E(|u|^2)E(|w|^2)$, then

$$\sigma_p(M_wEM_u) = \sigma_{jp}(M_wEM_u).$$

Proof. Let $f \in L^2(\Sigma) \setminus \{0\}$ and $\lambda \in \mathbb{C}$, such that $wE(uf) = \lambda f$. Let $M = span\{f\}$, the closed linear subspace generated by f. Thus we can represent $T = M_wEM_u$ as the following 2×2 operator matrix with respect to the decomposition $L^2(\Sigma) = M \oplus M^\perp$,

$$T = \begin{bmatrix} M_\lambda & PM_wEM_u - M_\lambda \\ 0 & M_wEM_u - PM_wEM_u \end{bmatrix}.$$

where P is is the orthogonal projection of $L^2(\Sigma)$ onto M. Since $|E(uw)|^2 = E(|u|^2)E(|w|^2)$, then we have $P(|T^2| - |T^*|^2)P \geq 0$. Direct computation shows that

$$P|T^2|^2P = \begin{bmatrix} M_{|\lambda|^4} & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} M_{|\lambda|^2} & 0 \\ 0 & 0 \end{bmatrix} = (P|T^2|^2P)^{\frac{1}{2}} \geq P|T^2|P \geq P|T^*|^2P$$

$$= PTT^*P = \begin{bmatrix} M_{|\lambda|^2} + AA^* & 0 \\ 0 & 0 \end{bmatrix},$$

where $A = PM_wEM_u - M_\lambda$. This implies that $A = 0$ and so $T^*(f) = M_{\bar{u}}EM_{\bar{w}}(f) = \bar{\lambda}f$. This means that $\sigma_p(M_wEM_u) = \sigma_{jp}(M_wEM_u)$.

Corollary 3.8. If $T = M_wEM_u$ is p-hyponormal and $S(E(u)) = X$, then

(1) $\sigma_p(M_wEM_u) = \sigma_{jp}(M_wEM_u)$.

(2) $\sigma_{jp}(M_wEM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} : \mu(A_{\lambda,w}) > 0\} \setminus \{0\}$,
where $A_{\lambda,w} = \{x \in X : E(uw)(x) = \lambda\}$.

4. Examples

In this section we present some examples of conditional expectations and corresponding multiplication operators to illustrate concrete application of the main results of the paper.

Example 4.1. Let $X = [0,1]$, Σ=sigma algebra of Lebesgue measurable subset of X, μ=Lebesgue measure on X. Fix $n \in \{2,3,4...\}$ and let $s : [0,1] \to [0,1]$ be defined by $s(x) = x + \frac{1}{n} (\mathrm{mod}\ 1)$. Let $\mathcal{B} = \{E \in \Sigma : s^{-1}(E) = E\}$. In this case $E^{\mathcal{B}}(f)(x) = \sum_{j=0}^{n-1} f(s^j(x))$, where s^j denotes the jth iteration of s. The functions f in the range of $E^{\mathcal{B}}$ are those for which the n graphs of f restricted to the intervals $[\frac{j-1}{n}, \frac{j}{n}]$, $1 \leq j \leq n$, are all congruent. Also, $|f| \leq nE^{\mathcal{B}}(|f|)$ a.e. Hence, the operator EM_u is bounded on $L^2([0,1])$ if and only if $u \in L^\infty([0,1])$. These operators are closely related to averaging operators.

Example 4.2. Let $X = [0,1] \times [0,1]$, $d\mu = dxdy$, Σ the Lebesgue subsets of X and let $\mathcal{A} = \{A \times [0,1] : A$ is a Lebesgue set in $[0,1]\}$. Then, for each f in $L^2(\Sigma)$, $(Ef)(x,y) = \int_0^1 f(x,t)dt$, which is independent of the second coordinate. Now, if we take $u(x,y) = y^{\frac{x}{8}}$ and $w(x,y) = \sqrt{(4+x)y}$, then $E(|u|^2)(x,y) = \frac{4}{4+x}$ and $E(|w|^2)(x,y) = \frac{4+x}{2}$. So, $E(|u|^2)(x,y)E(|w|^2)(x,y) = 2$ and $|E(uw)|^2(x,y) = 64\frac{4+x}{(x+12)^2}$. Direct computations shows that $E(|u|^2)(x,y)E(|w|^2)(x,y) \leq |E(uw)|^2(x,y)$, a.e, and since the operator M_wEM_u is bounded, then $E(|u|^2)(x,y)E(|w|^2)(x,y) \geq |E(uw)|^2(x,y)$ a.e. Thus, $E(|u|^2)(x,y)E(|w|^2)(x,y) = |E(uw)|^2(x,y)$, a.e. Therefore by Theorem 3.7 we conclude that

$$\sigma_p(M_wEM_u) = \sigma_{jp}(M_wEM_u) = \{\sqrt{2}\}.$$

Example 4.3. Let $X = [0,1)$, Σ=sigma algebra of Lebesgue measurable subset of X, μ=Lebesgue measure on X. Let $s : [0,1) \to [0,1)$ be defined by $s(x) = x + \frac{1}{2} (\mathrm{mod}\ 1)$. Let $\mathcal{B} = \{E \in \Sigma : s(E) = E\}$. In this case

$$E^{\mathcal{B}}(f)(x) = \frac{f(x) + f(s(x))}{2}.$$

Also, $|f| \leq E^{\mathcal{B}}(|f|)$ a.e. Hence, the operator EM_u is bounded on $L^2([0,1))$ if and only if $u \in L^{\infty}([0,1))$. Let $u(x) = \sqrt{x}$ and $w \equiv 1$, then $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$ a.e. Thus by Theorem 3.7 we have

$$\sigma_p(M_w EM_u) = \sigma_{jp}(M_w EM_u) = \emptyset.$$

Example 4.4. Let $X = \mathbb{N}$, $\mathcal{G} = 2^{\mathbb{N}}$ and let $\mu(\{x\}) = pq^{x-1}$, for each $x \in X$, $0 \leq p \leq 1$ and $q = 1 - p$. Elementary calculations show that μ is a probability measure on $\mathcal{G}$. Let $\mathcal{A}$ be the σ-algebra generated by the partition $B = \{X_1, X_1^c\}$ of X, where $X_1 = \{3n : n \geq 1\}$. So, for every $f \in \mathcal{D}(E^{\mathcal{A}})$,

$$E^{\mathcal{A}}(f) = \alpha_1 \chi_{X_1} + \alpha_2 \chi_{X_1^c}$$

and direct computations show that

$$\alpha_1 = \frac{\sum_{n\geq 1} f(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{3n-1}}$$

and

$$\alpha_2 = \frac{\sum_{n\geq 1} f(n)pq^{n-1} - \sum_{n\geq 1} f(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{n-1} - \sum_{n\geq 1} pq^{3n-1}}.$$

For example, if we set $f(x) = x$, then $E^{\mathcal{A}}(f)$ is a special function as follows;

$$E^{\mathcal{A}}(f) = \alpha_1 \chi_{X_1} + \alpha_2 \chi_{X_1^c}$$

where

$$\alpha_1 = \frac{3}{1 - q^3}, \quad \alpha_2 = \frac{1 + q^6 - 3q^4 + 4q^3 - 3q^2}{(1 - q^2)(1 - q^3)}.$$

So, if u and w are real functions on X such that $M_w EM_u$ is boundede on l^p, then

$$\begin{aligned}\sigma(M_w EM_u) &= \sigma_p(M_w EM_u)\\ &= \{\frac{\sum_{n\geq 1} u(3n)w(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{3n-1}}, \frac{\sum_{n\geq 1} u(n)w(n)pq^{n-1} - \sum_{n\geq 1} u(3n)w(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{n-1} - \sum_{n\geq 1} pq^{3n-1}}\}.\end{aligned}$$

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