

THE SPECTRAL FUNCTION OF VARIOUS DIFFERENTIAL OPERATORS AND ITS ASYMPTOTICAL BEHAVIOUR

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ABSTRACT. *In this paper we shall consider one type of spectral functions which is characteristic for discrete operators, given by ordinary differential operators on bounded segments.*

1. One of the important problems in the spectral theory of differential operators is the investigation of their spectral functions and, particularly, the asymptotic behaviour of such functions for large values of the spectral parameter. Here, by "spectral function" one can mean various functions, all closely related with the spectral resolution of the given operator. In this paper we shall consider one type of spectral functions which is characteristic for discrete operators, given by ordinary differential operators on bounded segments.

As a simple example, consider the Sturm-Liouville operator

$$(1) \quad -y'' + q(x)y = \lambda y, \quad y'(0) - hy(0) = y'(\pi) + Hy(\pi),$$

where q is a real, sufficiently smooth, function and $h, H \in \mathbb{R}$. For such an operator the spectral function is given by

$$\rho(\lambda) = \sum_{\lambda_n < \lambda} \frac{1}{\alpha_n},$$

where λ_n are the eigenvalues, y_n the corresponding eigenfunctions of the operator (1), chosen in such a way that $y_n(0) = 1$ and $\alpha_n = \int_0^\pi y_n^2(x) dx$. The problem of the investigation of the asymptotic behaviour of $\rho(\lambda)$ when $\lambda \rightarrow \infty$ was considered for a long time. One of the simplest answer was given by V.A.Sadovnichii in [7] by proving the following.

THEOREM 1 *If $\tilde{\rho}$ is the spectral function of the operator of the type (1), where the function q is replaced by \tilde{q} , then*

$$(2) \quad \tilde{\rho}(\lambda) = \rho(\lambda) + o(1), \quad \lambda \rightarrow \infty.$$

The proof of the theorem was based on calculating sums of the type

$$(3) \quad \sum_n [z_n^m \beta_n - A_m(n, \beta_n)] = B_m,$$

where z_n are zeros of a certain entire function $f(z)$, m is a positive integer, β_n certain "weights" and $A_m(n, \beta_n)$ completely determined numbers which enable the convergence of the series. Here the function $f(z)$ is closely related with the given operator and its structure is strictly determined. When the operator (1) is concerned, the formula (3) gives

$$(4) \quad \sum_n \left(\frac{1}{\alpha_n} - \frac{2}{\pi} \right) = \frac{1}{\pi},$$

and the formula (2) follows easily.

A natural question is whether similar results can be derived for other types of differential boundary value problems. Recall that even in the Tamarkin's paper [8] a general variant of such problem was given, where generalizations were: 1° the differential expression could be of higher order; 2° the spectral parameter could be contained in boundary conditions; 3° boundary conditions could contain some integrals; 4° the differential expression could polynomially depend on the spectral parameter. Several examples illustrating the cases 1°–4°, as well as some problems of different types, were considered and the corresponding answer to our question was given in the papers [1]–[5]. Here we shall discuss only some of the characteristic problems which arose during the investigation of these operators.

2. One of the main problems is the way of choosing the eigenfunctions y_n which define the coefficients α_n and the function ρ . Such a problem arises even for the Sturm-Liouville operator, but with boundary conditions different from (1):

$$(5) \quad -y'' + q(x)y = \lambda y, \quad y(0) = y(\pi) = 0$$

Here one cannot choose y_n so that $y_n(0) = 1$. If they are chosen by the conditions $y'_n(0) = 1$, however, it is shown in [2] that the formula (2) is not valid anymore in the general case. Only the following weaker formulation is possible:

THEOREM 2 *If ρ and $\tilde{\rho}$ are spectral functions of the operators of the type (5) (the former with the function q and the latter with \tilde{q}) and if $q(0) = \tilde{q}(0)$, $q'(0) = \tilde{q}'(0)$, then the formula (2) is valid.*

Namely, instead of (4) one can get the formula

$$\sum_n \left[\frac{1}{\alpha_n} - \frac{2}{\pi} n^2 + \frac{1}{\pi} q(0) \right] = -\frac{1}{4} q'(0) - \frac{1}{2\pi} q(0).$$

Nevertheless, it is proved that the formula (2) is possible without further assumptions on q and \tilde{q} in the case when the choice of the eigenfunctions y_n is such that it

depends on the "potential". For example, if we put $y'_n(0) = s_n$, $s_n^2 = \lambda_n$, we obtain the formula (4) and so also (2).

The similar effect appears in the cases considered in [1]–[5]. The main problem there is to find the way of choosing y_n 's which can ensure obtaining the formula (2).

3. When nonselfadjoint differential or functional-differential boundary value problems are concerned, the basic question is to define the function $\rho(\lambda)$ which would naturally bare the name "spectral function". The question is answered depending on the known formulas for the corresponding spectral resolutions. As an example, consider the Regge operator

$$-y'' + q(x)y = s^2y, \quad y(0) = y'(a) + isy(a) = 0,$$

where q is a complex function, $q(x) \sim c_\mu(a-x)^\mu$ ($x \rightarrow a-0$), $\mu \geq 0$, $c_\mu \neq 0$. In the paper [6] it was shown that the eigenvalues s_n behave like

$$s_n = \frac{n\pi}{a} + i\frac{\mu+2}{2a} \ln|n| + O(1) \quad (n \rightarrow \infty),$$

and that the functions f and g , satisfying certain conditions, can be written in the form $f(x) = \sum_{-\infty}^{\infty} c_n y_n(x)$, $g(x) = \sum_{-\infty}^{\infty} c_n s_n y_n(x)$, where

$$c_n = \frac{1}{2} \frac{\int_0^a [f(x) + g(x)/s_n] y_n(x) dx - i y_n(a) f(a)/s_n}{\int_0^a y_n^2(x) dx - i y_n^2(a)/2s_n}.$$

So, it is possible to define the spectral function by

$$\rho(\lambda) = \sum_{|\operatorname{Re} s_n| < \lambda} \frac{1}{\alpha_n}, \quad \alpha_n = \int_0^a y_n^2(x) dx - i \frac{y_n^2(a)}{2s_n}.$$

For such a function, the theorems of the type 1 and 2 are proved in [1].

Let us mention that the corresponding construction can sometimes be made even when formulas for spectral resolution are unknown. E. g. it was done in [1] for the case of the Orr-Sommerfeld equation, using some considerations in Tamarkin's work [8]. (The corresponding resolution was proved later).

4. As a special problem when calculating the sums of the type (3) for differential operators of various kinds, one have to consider the shape of the indicator diagram of corresponding characteristic function. Namely, in the example given in [7], such a diagram was a polygon without "spare" pints on its edges. When some other operators are concerned, this may not be the case. As an example consider the problem

$$(6) \quad -y'' + q(x)y = \lambda y, \quad y(0) = \int_0^\pi y(x) \mu(x) dx, \quad y(\pi) = 0,$$

where q and μ are real functions, $\mu(x) \geq 0$, $\int_0^\pi \mu(x) dx = 1$, $\int_0^\pi q(x) dx = 0$. Here (see[3]) the indicator diagram is $\{i\pi, 0, -i\pi\}$ which causes modifications of the method of calculating the sums that are needed. These calculations are too long to be given here. As a final result, instead of (4) one gets

$$\sum_n \left\{ \frac{1}{\alpha_n} - \frac{1}{\pi}(4n^2 + D_1) - \frac{1}{\pi} \left[4 \left(n + \frac{1}{2} \right)^2 + D_2 \right] \right\} = E,$$

where the constants D_1 , D_2 and E depend on the values of q and μ in 0 and π . The similar method is used in the papers [1], [3] and [5].

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