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ON A REPRESENTATION THEOREM FOR SPACES A^{φ}_{sr}

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ABSTRACT. *In* [2] *F. Ricci and M. Taibleson have obtained representation theorems for spaces* $A^{\beta}_{sr} = A^{y^{\beta}}_{sr}$, $\beta > 0$. *By an adaptation of the proof given in* [2] *we will extend their results to the spaces* A^{φ}_{sr}.

1. Introduction. A positive continuous function φ on $(0,\infty)$ is almost increasing if there exists a positive constant C such that $x_1 < x_2$ implies $\varphi(x_1) \le C\varphi(x_2)$. An almost decreasing function is defined similarly. A positive continuous increasing function φ is normal on $(0,\infty)$ if there exist constants a, b, $0 < a < b$ such that

$$\frac{\varphi(x)}{x^a} \quad \text{is almost increasing} \tag{1.1}$$

and

$$\frac{\varphi(x)}{x^b} \quad \text{is almost decreasing.} \tag{1.2}$$

Throughout this paper φ will denote a normal function satisfying (1.1) and (1.2).

Suppose $0 < s \le \infty$ and that $f(z)$ is a function holomorphic on the upper half-plane $R^2_+ = \{z = x + iy : x \in R,\, y > 0\}$.
Then, let

$$M_s(y,f) = \left(\int_{-\infty}^{+\infty} |f(x+iy|^s\,dx\right)^{1/s}, \qquad 0 < s < \infty,$$
$$M_\infty(y,f) = \sup_x |f(x+iy)|.$$

If $0 < s, r \le \infty$, a function f, holomorphic on R^2_+, is said to belong to the space A^{φ}_{sr} if

$$N^{\varphi}_{sr}(f) = \left(\int_0^\infty \varphi(y)^r M_s(y,f)^r \frac{dy}{y}\right)^{1/r} < \infty, \quad 0 < r < \infty,$$
$$N^{\varphi}_{s\infty}(f) = \sup_y \varphi(y) M_s(y,f) < \infty.$$

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Our main result is the following representation theorem for spaces A^{φ}_{sr}.

THEOREM *Suppose* $0 < s, r \leq \infty, \eta > \max\{b + 1/s, b + 1\}$. *Then there is a collection of points* $\{\xi^k_{lj}\}$ *in* R^2_+ *so that*

i) *If* $\lambda = \{\lambda^k_{lj}\}$ *is a sequence of complex numbers such that* $||\lambda||_{sr} < \infty$ *then the series*

$$\sum_{ljk} \lambda^k_{lj} \frac{(\operatorname{Im} \xi^k_{lj})^{\eta - 1/s}}{\varphi(\operatorname{Im} \xi^k_{lj})(z - \overline{\xi^k_{lj}})^{\eta}}$$

converges absolutely and uniformly on compact subsets of R^2_+ *to a holomorphic function* f *in* A^{φ}_{sr} *and there is a constant* $C > 0$ *which depends only on* s, r, a, b, η *and* M *so that* $N^{\varphi}_{sr}(f) \leq C||\lambda||_{sr}$.

ii) *If* $f \in A^{\varphi}_{sr}$ *then there is a sequence* $\lambda = \{\lambda^k_{lj}\}, ||\lambda||_{sr} < \infty$ *such that*

$$f(x + iy) = \sum_{ljk} \lambda^k_{lj} \frac{(\operatorname{Im} \xi^k_{lj})^{\eta - 1/s}}{\varphi(\operatorname{Im} \xi^k_{lj})(z - \overline{\xi^k_{lj}})^{\eta}}$$

and there is a constant $C > 0$ *which depends only on* s, r, a, b, η *and* M *such that* $||\lambda||_{sr} \leq CN^{\varphi}_{sr}(f)$.

2. Preliminaries.

LEMMA 2.1 *There exist* $C > 0$ *such that for all* $x > 0$, $\varphi(2x) \leq C\varphi(x)$.

PROOF. By assumption $\varphi(x)/x^b$ is almost decreasing. Hence,

$$\frac{\varphi(2x)}{\varphi(x)} = \frac{\varphi(2x)}{(2x)^b} \frac{x^b}{\varphi(x)} 2^b \leq c2^b = C.$$

Throughout this paper c and C denote positive constants not necessarily the same at each occurrence.

PROPOSITION 2.2 *Suppose* $0 < s \leq s_1 \leq \infty$, $0 < r \leq r_1 \leq \infty$. *Then* $A^{\varphi}_{sr} \subset A^{\varphi}_{sr_1}$, $A^{\varphi}_{sr} \subset A^{\varphi y^{1/s - 1/s_1}}_{s_1 r}$, *and the inclusions are continuous.*

PROOF. From Lemma 2.1 ([2],p.5) we have that if $f \in A^{\varphi}_{sr}$ then

$$M_s(y, f) \leq C \left(\int_{y/2}^{3y/2} M_s(t, f)^r \frac{dt}{t} \right)^{1/r}.$$

By using Lemma 2.1 we find that

$$\varphi(y) M_s(y, f) \leq C\varphi(\frac{y}{2}) \left(\int_{y/2}^{3y/2} M_s(t, f)^r \frac{dt}{t} \right)^{1/r} \leq$$

$$\leq C \left(\int_{y/2}^{3y/2} \varphi(t)^r M_s(t, f)^r \frac{dt}{t} \right)^{1/r} \leq CN^{\varphi}_{sr}(f),$$

and consequently $N^{\varphi}_{s\infty}(f) \le CN^{\varphi}_{sr}(f)$.
If $r < r_1 < \infty$, then

$$\begin{aligned}
N^{\varphi}_{sr_1}(f) &= \left(\int_0^{\infty} \varphi(y)^{r_1} M_s(y,f)^{r_1} \frac{dy}{y}\right)^{1/r_1} = \\
&= \left(\int_0^{\infty} \varphi(y)^{r_1} M_s(y,f)^{r_1-r} M_s(y,f)^{r} \frac{dy}{y}\right)^{1/r_1} \le \\
&\le C\left[\int_0^{\infty} \varphi(y)^{r_1} \left(\frac{N^{\varphi}_{sr}(f)}{\varphi(y)}\right)^{r_1-r} M_s(y,f)^{r} \frac{dy}{y}\right]^{1/r_1} = \\
&= C\,(N^{\varphi}_{sr}(f))^{1-r/r_1}\,(N^{\varphi}_{sr}(f))^{r/r_1} = CN^{\varphi}_{sr}(f).
\end{aligned}$$

If $0 < s < \infty$, then

$$M_{\infty}(y,f) \le C\left(\frac{1}{y}\int_{y/2}^{3y/2} M_s(t,f)^{s}\frac{dt}{t}\right)^{1/s} \qquad \text{see ([2], p. 6)}$$

If $0 < s \le r$, we have

$$\begin{aligned}
N^{\varphi y^{1/s}}_{\infty r}(f) &= \left(\int_0^{\infty} \varphi(y)^{r} y^{r/s} M_{\infty}(y,f)^{r} \frac{dy}{y}\right)^{1/r} \le \\
&\le C\left[\int_0^{\infty} \varphi(y)^{r} \left(\int_{y/2}^{3y/2} M_s(t,f)^{s}\frac{dt}{t}\right)^{r/s} \frac{dy}{y}\right]^{1/r} \le \\
&\le C\left[\int_0^{\infty} \varphi(y)^{r} \left(\int_{y/2}^{3y/2} M_s(t,f)^{r}\frac{dt}{t}\right) \frac{dy}{y}\right]^{1/r} = \\
&= C\left(\int_0^{\infty} M_s(t,f)^{r}\frac{dt}{t}\int_{2t/3}^{2t} \varphi(y)^{r}\frac{dy}{y}\right)^{1/r} \le \\
&\le C\left(\int_0^{\infty} \varphi(t)^{r} M_s(t,f)^{r}\frac{dt}{t}\right)^{1/r} = CN^{\varphi}_{sr}(f).
\end{aligned}$$

If $r < s \le \infty$, then

$$M_{\infty}(y,f) \le Cy^{-1/s}\left(\int_{y/2}^{3y/2} M_s(t,f)^{r}\frac{dt}{t}\right)^{1/r}$$

and the rest of the proof goes just as for the case $0 < s \le r$.
Let now $s < s_1 < \infty$. Then

$$N^{\varphi y^{1/s-1/s_1}}_{s_1 r}(f) = \left(\int_0^{\infty} \varphi(y)^{r} y^{(1/s-1/s_1)r} M_{s_1}(y,f)^{r}\frac{dy}{y}\right)^{1/r} \le$$

$$\leq \left[\int_0^\infty \varphi(y)^r y^{(1/s-1/s_1)r} M_\infty(y,f)^{(s_1-s)r/s_1} M_s(y,f)^{sr/s_1} \frac{dy}{y}\right]^{1/r}.$$

Hölder inequality with $\alpha = s_1/s$, $\alpha' = s_1/(s_1-s)$ gives $N_{s_1 r}^{\varphi y^{1/s-1/s_1}}(f) \leq$

$$\leq \left[\int_0^\infty \varphi(y)^r y^{r/s} M_\infty(y,f)^r \frac{dy}{y}\right]^{(1-s/s_1)/r} \left[\int_0^\infty \varphi(y)^r M_s(y,f)^r \frac{dy}{y}\right]^{s/(rs_1)} =$$

$$C\left[N_{\infty r}^{\varphi y^{1/s}}(f)\right]^{1-s/s_1} [N_{sr}^\varphi(f)]^{s/s_1} \leq C\,[N_{sr}^\varphi(f)]^{1-s/s_1} [N_{sr}^\varphi(f)]^{s/s_1} = CN_{sr}^\varphi(f).$$

PROPOSITION 2.3 *If $f \in A_{sr}^\varphi$ and $\eta > \max\{b+1/s,\, b+1\}$ then*

$$f(z) = C_\eta \int\limits_{R_+^2} f(\xi)\frac{(\operatorname{Im}\xi)^{\eta-2}}{(z-\bar{\xi})^\eta}\, d\xi$$

for each $z \in R_+^2$, and the integral converges absolutely.

PROOF. We may proceed as in the proof of Lemma 4.4 ([2]) using Proposition 2.2 instead of Proposition 2.2 ([2], p. 5).

3. The representation theorems. We will use the following lemma:

LEMMA 3.1 *Let $\eta > b+1$, $1 < s < \infty$, $1/s + 1/s' = 1$. Then there exist constants θ, τ, $0 < \theta, \tau < 1$, such that the following conditions are satisfied:*

(i) $\eta(1-\theta)s' > 1$

(ii) $\eta\theta s > 1$

(iii) $(\eta - 1/s)(1-\tau) > b(1-\tau) + 1/s'$

(iv) $(\eta - 1/s)(1-\tau) > \eta(1-\theta) - a\tau$

(v) $(\eta - 1/s)(1-\tau) < \eta(1-\theta) + a(1-\tau)$

PROOF. We may rewrite the conditions of the Lemma as follows:

(i) $$\eta(1-\theta)s' > 1 \quad \text{iff} \quad \theta < 1 - \frac{1}{\eta s'}$$

(ii) $$\eta\theta s > 1 \quad \text{iff} \quad \theta > \frac{1}{\eta s}$$

(iii) $$(\eta - \frac{1}{s})(1-\tau) > b(1-\tau) + \frac{1}{s'} \quad \text{iff} \quad \tau < \frac{\eta-(b+1)}{\eta-(b+1/s)}$$

(iv) $$(\eta - \frac{1}{s})(1-\tau) > \eta(1-\theta) - a\tau \quad \text{iff} \quad \theta > \frac{(\eta-a)\tau + (1-\tau)/s}{\eta}$$

(v) $$(\eta - \frac{1}{s})(1-\tau) < \eta(1-\theta) + a(1-\tau) \quad \text{iff} \quad \theta < \frac{\eta\tau + (1-\tau)(a+1/s)}{\eta}.$$

Since

$$\frac{\eta-(b+1)}{\eta-(b+1/s)} < \frac{\eta-(a+1)}{\eta-(a+1/s)}, \quad 0<a<b,$$

then we see that if τ satisfies (iii) then $\tau < \frac{\eta-(a+1)}{\eta-(a+1/s)}$. The last condition is equivalent to $\frac{\eta\tau+(1-\tau)(a+1/s)}{\eta-(b+1/s)} < 1-\frac{1}{s'\eta}$. So, we may eliminate (i). From (iv) we have (ii) because $\frac{1}{s\eta} < \frac{\tau(\eta-a)+(1-\tau)/s}{\eta}$ iff $\eta > a+1/s$. Thus, these conditions need to be satisfied: (iii), (iv), and (v). An easy computation shows that (iv) is equivalent to $\tau < \frac{\eta\theta-1/s}{\eta-(a+1/s)}$ and (v) to $\tau > \frac{\eta\theta-(a+1/s)}{\eta-(a+1/s)}$. If we choose, for example, $0<\theta=\frac{a+1/s}{\eta}<1$, then the last condition is satisfied for all τ, $0<\tau<1$. Then we choose $\tau < \min\{\frac{a}{\eta-(a+1/s)}, \frac{\eta-(b+1)}{\eta-(b+1/s)}\}$.

Suppose $0<s,r\le\infty$ and that $\lambda=\{\lambda^k_{lj}\}, l,j\in Z, 1\le k\le M^2$, ($M$ is a positive integer) is a sequence of complex numbers. Then, with the usual convention for s or $r=\infty$, let $||\lambda||_{sr}=\left[\sum_j\left(\sum_{lk}|\lambda^k_{lj}|^s\right)^{r/s}\right]^{1/r}$.

LEMMA 3.2 *Suppose we are given $\lambda=\{\lambda_{lj}\}, l,j\in Z, \lambda_{lj}\ge 0, 0<s,r\le\infty$ and $\eta>\max\{b+1,b+1/s\}$ with $||\lambda||_{sr}<\infty$. Then let*

$$f(x,y)=\sum_{lj}\lambda_{lj}\frac{2^{j(\eta-1/s)}}{\varphi(2^j)[(x-l2^j)^2+(y+2^j)^2]^{\eta/2}}$$

for $(x,y)\in R^2_+$. Then the series defining f converges uniformly on compact subsets of R^2_+ and $N^{\varphi}_{sr}(f)\le C||\lambda||_{sr}$, where the constant C depends only on η,s,r,a and b.

PROOF. If $0<s\le 1$, then the series converges uniformly on each proper subhalfplane $\{(x,y)\in R^2_+ : y\ge y_0\}$ provided $\eta>b+1/s$. If $1<s\le\infty$ and $\eta>b+1$, the series converges uniformly on the regions of the form $\{(x,y)\in R^2_+ : y\ge y_0>0, |x|\le x_0<\infty\}$.

Estimates for $\mathbf{N^{\varphi}_{sr}(f)}$. In this part of the proof we first consider the case $0<s\le 1$ with subcases $r=\infty, s\le r<\infty$ and $0<r\le s$.

We have $$f(x,y)^s \le \sum_j\frac{2^{j(\eta-1/s)s}}{\varphi(2^j)^s}\sum_l\lambda^s_{lj}\frac{1}{[(x-l2^j)^2+(y+2^j)^2]^{\eta s/2}}.$$

Thus,

$$\varphi(y)^s M_s(y,f)^s \le C\sum_j A_j(y)\sum_l\lambda^s_{lj}, \tag{3.1}$$

where $A_j(y) = \dfrac{2^{j(\eta-1/s)s}\varphi(y)^s}{\varphi(2^j)^s(y+2^j)^{\eta s-1}}$.

Case $r = \infty$.

$$\tag{3.2}\begin{aligned}\sum_{2^j\le y} A_j(y)\sum_l \lambda_{lj}^s &\le C\|\lambda\|_{s\infty}^s \frac{\varphi(y)^s}{y^{\eta s-1}}\sum_{2^j\le y}\frac{2^{j(\eta-b-1/s)s}2^{jbs}}{\varphi(2^j)^s} \le \\ &\le C\|\lambda\|_{s\infty}^s y^{1-\eta s+bs}\sum_{2^j\le y}2^{j(\eta-b-1/s)} \le C\|\lambda\|_{s\infty}^s,\end{aligned}$$

provided $\eta > b + 1/s$.

$$\tag{3.3}\begin{aligned}\sum_{2^j\ge y} A_j(y)\sum_l \lambda_{lj}^s &\le C\|\lambda\|_{s\infty}^s\varphi(y)^s\sum_{2^j\ge y}\frac{2^{-jsa}2^{jsa}}{\varphi(2^j)^s} \le \\ &\le C\|\lambda\|_{s\infty}^s y^{sa}\sum_{2^j\ge y}2^{-jsa} \le C\|\lambda\|_{s\infty}^s.\end{aligned}$$

From (3.1), (3.2) i (3.3) it follows $N_{s\infty}^{\varphi}(f) \le C\|\lambda\|_{s\infty}$.

Case $s \le r < \infty$.

$$\begin{aligned}[N_{sr}^{\varphi}(f)]^s &\le C\left[\int_0^\infty\left(\sum_j A_j(y)\sum_l\lambda_{lj}^s\right)^{r/s}\frac{dy}{y}\right]^{s/r} \le \\ &\le C\left[\sum_k\left(\sum_{j<k}A_j(2^k)\sum_l\lambda_{lj}^s\right)^{r/s}\right]^{s/r} + C\left[\sum_k\left(\sum_{j\ge k}A_j(2^k)\sum_l\lambda_{lj}^s\right)^{r/s}\right]^{s/r} \\ &\le C\left[\sum_k\left(\frac{2^{kbs}}{2^{k(\eta s-1)}}\sum_{j<k}2^{j(\eta s-1-bs)}\sum_l\lambda_{lj}^s\right)^{r/s}\right]^{s/r} + \\ &+ C\left[\sum_k\left(2^{kas}\sum_{j\ge k}2^{-jas}\sum_l\lambda_{lj}^s\right)^{r/s}\right]^{s/r} \le C\|\lambda\|_{sr}^s,\end{aligned}$$

where we needed $\eta > b + 1/s$.

Case: $0 < r \le s < 1$.

$$\tag{3.4}\begin{aligned}N_{sr}^{\varphi}(f)^r &\le C\int_0^\infty\left(\sum_j A_j(y)\sum_l\lambda_{lj}^s\right)^{r/s}\frac{dy}{y} \le \\ &\le C\sum_j\frac{2^{jr(\eta-1/s)}}{\varphi(2^j)^r}\left(\sum_l\lambda_{lj}^s\right)^{r/s}\int_0^\infty\frac{\varphi(y)^r}{(y+2^j)^{\eta r-r/s}}\frac{dy}{y}\end{aligned}$$

$$\int_0^{2^j} \frac{\varphi(y)^r}{(y+2^j)^{\eta r - r/s}} \frac{dy}{y} \le \frac{\varphi(2^j)^r}{2^{jr(\eta+a-1/s)}} \int_0^{2^j} y^{ar-1}\, dy \le \le C \frac{\varphi(2^j)^r}{(y+2^j)^{jr(\eta-1/s)}} \tag{3.5}$$

$$\int_{2^j}^{\infty} \frac{\varphi(y)^r}{(y+2^j)^{\eta r - r/s}} \frac{dy}{y} \le \frac{\varphi(2^j)^r}{2^{jbr}} \int_{2^j}^{\infty} y^{br-\eta r-1+r/s}\, dy \le C \frac{\varphi(2^j)^r}{2^{j(\eta r - r/s)}}, \tag{3.6}$$

provided $\eta > b + 1/s$.

From (3.4), (3.5) and (3.6) we have $N^{\varphi}_{sr}(f) \le C||\lambda||_{sr}$.

Let now $1 < s < \infty$ and let s' be the index conjugate to s. Choose $0 < \theta, \tau < 1$ so that the conditions of Lemma 3.1 are satisfied. Now we have

$$f(x,y)^s \le \sum_{lj} \lambda_{lj}^s \frac{2^{j(\eta-1/s)\tau s}}{\varphi(2^j)^{\tau s}[(x-l2^j)^2+(y+2^j)^2]^{\eta\theta s/2}} \cdot \left(\sum_{lj} \frac{2^{j(\eta-1/s)(1-\tau)s'}}{\varphi(2^j)^{(1-\tau)s'}[(x-l2^j)^2+(y+2^j)^2]^{\eta(1-\theta)s'/2}} \right)^{s/s'} = S_1 \cdot S_2^{s/s'} \tag{3.7}$$

If $\eta(1-\theta)s' > 1$ (condition (i) of Lemma 3.1) then

$$S_2 \le C \sum_j \frac{2^{j(\eta-1/s)(1-\tau)s'-1}}{\varphi(2^j)^{(1-\tau)s'}(y+2^j)^{\eta(1-\theta)s'-1}} = C \sum_j \frac{2^{jA}}{\varphi(2^j)^{(1-\theta)s'}(y+2^j)^B} \tag{3.8}$$

$$\sum_{2^j<y} \frac{2^{jA}}{\varphi(2^j)^{(1-\tau)s'}(y+2^j)^B} \le C \frac{y^{b(1-\tau)s'-B}}{\varphi(y)^{(1-\tau)s'}} \sum_{2^j<y} 2^{j(A-b(1-\tau)s')} \le \le C \frac{y^{A-B}}{\varphi(y)^{(1-\tau)s'}}, \tag{3.9}$$

provided only $A - b(1-\tau)s' > 0$.

Note that $A - b(1-\tau)s' > 0$ iff $(\eta - 1/s)(1-\tau) > b(1-\tau) + 1/s'$ (condition (ii) of Lemma 3.1)

$$\sum_{2^j\ge y} \frac{2^{jA}}{\varphi(2^j)^{(1-\tau)s'}(y+2^j)^B} \le C \frac{y^{(1-\tau)as'}}{\varphi(y)^{(1-\tau)s'}} \sum_{2^j\ge y} 2^{j(A-B-(1-\tau)as')} \le C \frac{y^{A-B}}{\varphi(y)^{(1-\tau)s'}}, \tag{3.10}$$

provided only $(\eta - 1/s)(1-\tau) < \eta(1-\theta) + a(1-\tau)$ (condition (v) of Lemma 3.1)

Combining (3.7);(3.8);(3.9) and (3.10) we obtain $\varphi(y)^s M_s(y,f)^s \le$

$$\begin{aligned}(3.11)\quad &\le C\sum_j \frac{2^{j(\eta-1/s)\tau s}\varphi(y)^{\tau s}}{\varphi(2^j)^{\tau s} y^{[\eta(1-\theta)-(\eta-1/s)(1-\tau)]s}(y+2^j)^{\eta\theta s-1}}\sum_l \lambda_{lj}^s = \\ &= C\sum_j\left(\sum_l \lambda_{lj}^s\right)B_j(y), \quad \text{where} \quad B_j(y)=\frac{\varphi(y)^{\tau s}}{\varphi(2^j)^{\tau s}}\cdot\frac{2^{jA_1}}{y^{B_1}(y+2^j)^{C_1}}.\end{aligned}$$

This requires $\eta\theta s > 1$ (condition (ii) of Lemma 3.1).

Note that $B_1 + C_1 = A_1$. We will consider three subcases:

Case $0 < r \le s$. Then $0 < r/s \le 1$, so

$$(3.12)\quad [N_{sr}^{\varphi}(f)]^r \le C\sum_j\left[\frac{2^{jA_1 r/s}}{\varphi(2^j)^{\tau r}}\left(\int_0^\infty \frac{\varphi(y)^{\tau r}}{y^{B_1 r/s}(y+2^j)^{C_1 r/s}}\frac{dy}{y}\right)\left(\sum_l \lambda_{lj}^s\right)^{r/s}\right]$$

$$\begin{aligned}(3.13)\quad &\int_0^{2^j}\frac{\varphi(y)^{\tau r}}{y^{B_1 r/s}(y+2^j)^{C_1 r/s}}\frac{dy}{y} \le C\frac{\varphi(2^j)^{\tau r}}{2^{ja\tau r}}\int_0^{2^j}\frac{y^{a\tau r}}{y^{B_1 r/s}(y+2^j)^{C_1 r/s}}\frac{dy}{y} = \\ &= C\frac{\varphi(2^j)^{\tau r}}{2^{jA_1 r/s}}\int_0^1 \frac{t^{a\tau r - B_1 r/s}}{(1+t)^{C_1 r/s}}\frac{dt}{t} \le C\frac{\varphi(2^j)^{\tau r}}{2^{jA_1 r/s}},\end{aligned}$$

provided only $(\eta - 1/s)(1-\tau) > \eta(1-\theta) - a\tau$ (condition (v) of Lemma 3.1)

$$\begin{aligned}(3.14)\quad &\int_{2^j}^\infty \frac{\varphi(y)^{\tau r}}{y^{B_1 r/s}(y+2^j)^{C_1 r/s}}\frac{dy}{y} \le C\frac{\varphi(2^j)^{\tau r}}{2^{jb\tau r}}\int_{2^j}^\infty \frac{y^{b\tau r - B_1 r/s}}{(y+2^j)^{C_1 r/s}}\frac{dy}{y} = \\ &= C\frac{\varphi(2^j)^{\tau r}}{2^{jA_1 r/s}}\int_1^\infty \frac{t^{b\tau r - B_1 r/s}}{(1+t)^{C_1 r/s}}\frac{dt}{t} \le C\frac{\varphi(2^j)^{\tau r}}{2^{jA_1 r/s}}.\end{aligned}$$

This requires $\eta > b + 1/s$.

From (3.12), (3.13) and (3.14) follows $N_{sr}^{\varphi}(f) \le C\|\lambda\|_{sr}$.

Case $r = \infty$. From (3.11) we have

$$(3.15)\quad N_{s\infty}^{\varphi}(f) \le C\|\lambda\|_{s\infty}\left(\sum_j B_j(y)\right)^{1/s} \le C\|\lambda\|_{s\infty},$$

(Note that the proof of (3.9) and (3.10) shows that $\sum_j \frac{2^{jA_1}}{\varphi(2^j)^{\tau s}(y+2^j)^{C_1}} \le$ $\le C\frac{y^{B_1}}{\varphi(y)^{\tau s}}$, provided : $\eta > b + 1/s$, $(\eta - 1/s)(1-\theta) > \eta(1-\theta) - a\tau$ all of which are satisfied).

Case $1 \le r/s < \infty$. From (3.11) we have again

$$[N^{\varphi}_{sr}(f)]^s \le C\left[\int_0^{\infty}\left(\sum_j B_j(y)\right)^{r/s}\frac{dy}{y}\right]^{s/r} \le C\left[\sum_k\left(\sum_{j\ge k}B_j(2^k)\right)^{r/s}\right]^{s/r}$$

$$+C\left[\sum_k\left(\sum_{j<k}B_j(2^k)\right)^{r/s}\right]^{s/r} \le C\left[\sum_k\left(\frac{2^{ka\tau s}}{2^{kB_1}}\sum_{j\ge k}2^{j(B_1-a\tau s)}\sum_l \lambda^s_{lj}\right)^{r/s}\right]^{s/r}$$

$$+C\left[\sum_k\left(\frac{2^{kb\tau s}}{2^{kA_1}}\sum_{j<k}2^{j(A_1-b\tau s)}\sum_l \lambda^s_{lj}\right)^{r/s}\right]^{s/r} \le C||\lambda||^s_{sr}.$$

This requires $(\eta-1/s)(1-\tau) > \eta(1-\theta)-a\tau$ and $\eta > b+1/s$, all of which are satisfied. Assume now $s=\infty$. Then we have

$$\begin{aligned} f(x,y) &= \sum_j \frac{2^{j\eta}}{\varphi(2^j)}\sum_l \frac{\lambda_{lj}}{[(x-l2^j)^2+(y+2^J)]^{\eta/2}} \le \\ &\le C\sum_j\left(\sup_l \lambda_{lj}\right)\frac{2^{j(\eta-1)}}{\varphi(2^j)(y+2^j)^{\eta-1}}. \end{aligned} \tag{3.16}$$

From (3.16) we have, if $0<r\le 1$,

$$[N^{\varphi}_{\infty r}(f)]^r \le C\sum_j\left(\sup_l \lambda_{lj}\right)^r \frac{2^{j(\eta-1)r}}{\varphi(2^j)^r}\int_0^{\infty}\frac{\varphi(y)^r}{(y+2^j)^{(\eta-1)r}}\frac{dy}{y} \le C||\lambda||^r_{\infty r}.$$

This requires $\eta > b+1$.
If $1<r<\infty$, then

$$[N^{\varphi}_{\infty r}(f)]^r \le C\left[\int_0^{\infty}\left(\sum_j \sup_l \lambda_{lj}\frac{2^{j(\eta-1)}}{\varphi(2^j)(y+2^j)^{\eta-1}}\right)^r\frac{dy}{y}\right] \le C||\lambda||^r_{\infty r},$$

provided only $\eta > b+1$.

Finally if $r=\infty$, then

$$N^{\varphi}_{\infty\infty}(f) \le C\sup_y\sum_j\left(\sup_l \lambda_{lj}\right)\frac{2^{j(\eta-1)}}{\varphi(2^j)(y+2^j)^{\eta 1-}} \le C||\lambda||_{\infty\infty},$$

provided $\eta > b+1$.

This finishes the proof of Lemma 3.2.

Divide R^2_+ into squares Q_{lj} with vertices $l2^j+i2^j, (l+1)2^j+i2^j, l2^j+i2^{j+1}$ and $(l+1)2^j+i2^{j+1}$. Then divide each square Q_{lj} into M^2 equal squares $Q^k_{lj}, k=1,2,\dots,M^2$, each of side length $2^j/M$.

LEMMA 3.3 *If f is holomorphic on R^2_+, then $N^{\varphi}_{sr}(f)$ is equivalent to*

$$\left\| \left\{ 2^{j/s} \varphi(2^j) \sup_{z \in Q_{lj}} |f(z)| \right\} \right\|_{sr} .$$

PROOF. For the sake of simplicity we will write the proof for the cases where $s, r \neq \infty$. Adjustments for the exceptional cases are trivial.

$$N^{\varphi}_{sr}(f) = \left[\sum_j \int_{2^j}^{2^{j+1}} \varphi(y)^r \left(\sum_l \int_{l2^j}^{(l+1)2^j} |f(x+iy)|^s \, dx \right)^{r/s} \frac{dy}{y} \right]^{1/r}$$

$$\leq C \left[\sum_j \int_{2^j}^{2^{j+1}} \varphi(y)^r \left(\sum_l 2^j \sup_{z \in Q_{lj}} |f(z)|^s \right)^{r/s} \frac{dy}{y} \right]^{1/r}$$

$$\leq C \left[\sum_j \varphi(2^j)^r 2^{jr/s} \left(\sum_l \sup_{z \in Q_{lj}} |f(z)|^s \right)^{r/s} \right]^{1/r} = C \left\| \left\{ \varphi(2^j)^{j/s} \sup_{z \in Q_{lj}} |f(z) \right\} \right\|_{sr} .$$

Conversely, by using subharmonicity of f we find:

$$\left[\sum_j \varphi(2^j)^r 2^{jr/s} \left(\sup_{z \in Q_{lj}} |f(z)|^s \right)^{r/s} \right]^{1/r} \leq C \left[\sum_j \varphi(2^j)^r \int_{2^{j-1}}^{2^{j+1}} M_s(y, f)^r \frac{dy}{y} \right]^{1/r}$$

$$\leq C N^{\varphi}_{sr}(f) .$$

4. Proof of Theorem. We choose $\{\xi^k_{lj}\}$ to be a collection of points in R^2_+ such that ξ^k_{lj} is any point in Q^k_{lj}.

To prove i) just observe that

$$\left| \sum_{k=1}^{M^2} \lambda^k_{lj} \frac{(\operatorname{Im} \xi^k_{lj})^{\eta - 1/s}}{\varphi(\operatorname{Im} \xi^k_{lj})(z - \overline{\xi^k_{lj}})^{\eta}} \right| \leq C \sum_{k=1}^{M^2} |\lambda^k_{lj}| \frac{2^{j(\eta - 1/s)}}{\varphi(2^j)[(x - l2^j)^2 + (y + 2^j)^2]^{\eta/2}}$$

$$\leq C_M \left(\sum_{k=1}^{M^2} |\lambda^k_{lj}|^s \right)^{1/s} \frac{2^{j(\eta - 1/s)}}{\varphi(2^j)[(x - l2^j)^2 + (y + 2^j)^2]^{\eta/2}} .$$

One then just applies Lemma 3.2 with $\lambda_{lj} = \left[\sum_{k=1}^{M^2} |\lambda^k_{lj}|^s \right]^{1/s}$.

It follows from the uniform convergence on compact subsets of R^2_+ that f is holomorphic and that $N^{\varphi}_{sr}(f) \leq C \|\lambda\|_{sr}$.

In order to prove the converse we will construct a sequence of functions $\{f_n\}$ in A^{φ}_{sr} such that

$$\text{(4.1)}\qquad \begin{aligned} &\text{i)}\quad f_n(z)=\sum_{ljk} C^{kn}_{lj}\frac{(\operatorname{Im}\xi^k_{lj})^{\eta-1/s}}{\varphi(\operatorname{Im}\xi^k_{lj})(z-\overline{\xi^k_{lj}})^{\eta}}\\ &\text{ii)}\quad \|C^n\|_{sr}\le C2^{-n}N^{\varphi}_{sr}(f)\quad\text{and so}\quad N^{\varphi}_{sr}(f_n)\le C2^{-n}N^{\varphi}_{sr}(f)\\ &\text{iii)}\quad N^{\varphi}_{sr}\left(f-\sum_{m=1}^{n} f_m\right)\le 2^{-n}N^{\varphi}_{sr}(f)\,, \end{aligned}$$

where we have set $C^n=\left\{C^{k\,n}_{lj}\right\}$. The result then follows easily.

Let $\lambda^k_{lj}=\sum_{n=1}^{\infty}C^{k\,n}_{lj}$. It follows from (4.1) ii) that $\|\lambda\|_{sr}\le CN^{\varphi}_{sr}(f)$.

Thus by the first part of this theorem $g(z)=\sum_{ljk}\lambda^k_{lj}\dfrac{(\operatorname{Im}\xi^k_{lj})^{\eta-1/s}}{\varphi(\xi^k_{lj})(z-\overline{\xi^k_{lj}})^{\eta}}$ is a function in A^{φ}_{sr}. But then we have that $N^{\varphi}_{sr}(g-\sum_{m=1}^{n}f_m)\le C\|\lambda-\sum_{m=1}^{n}C^m\|_{sr}\le C2^{-n}N^{\varphi}_{sr}(f)$.

Let $h=\min\{1,s,r\}$. Then for each positive integer n, $N^{\varphi}_{sr}(f-g)\le$

$$\le\left[N^{\varphi}_{sr}\left(f-\sum_{m=1}^{n}f_m\right)^{h}+N^{\varphi}_{sr}\left(g-\sum_{m=1}^{n}f_m\right)^{h}\right]^{1/h}\le 2^{-n}(1+C^h)^{1/h}N^{\varphi}_{sr}(f)\,.$$

Thus, $N^{\varphi}_{sr}(f-g)=0$ and so $f=g$.

The existence of a sequence $\{f_n\}$ follows from the corresponding properties of an operator S. Given $f\in A^{\varphi}_{sr}$ there is a function $Sf\in A^{\varphi}_{sr}$ such that

$$\text{(4.2)}\qquad \begin{aligned} &\text{i)}\quad Sf(z)=\sum_{ljk} C^{k}_{lj}\frac{(\operatorname{Im}\xi^k_{lj})^{\eta-1/s}}{\varphi(\operatorname{Im}\xi^k_{lj})(z-\overline{\xi^k_{lj}})^{\eta}}\\ &\text{ii)}\quad \|C\|_{sr}\le CN^{\varphi}_{sr}(f)\quad\text{and so}\quad N^{\varphi}_{sr}(Sf)\le CN^{\varphi}_{sr}(f)\\ &\text{iii)}\quad N^{\varphi}_{sr}(f-Sf)\le \frac{1}{2}N^{\varphi}_{sr}(f)\,. \end{aligned}$$

One then just lets $f_1=Sf$ and $f_n=S(f-\sum_{m=1}^{n-1}f_m)$ for $n>1$.

$$\text{For } f\in A^{\varphi}_{sr} \text{ we let}\qquad Sf(z)=\sum_{ljk}f(\xi^k_{lj})\frac{(\operatorname{Im}\xi^k_{lj})^{\eta-2}}{(z-\overline{\xi^k_{lj}})^{\eta}}|Q^k_{lj}|\,.$$

Note that $Sf(z)$ is the Rieman sum of f (Proposition 2.3) corresponding to the partition $\{Q^k_{lj}\}$ and the selection of points $\{\xi^k_{lj}\}$ (we will write out the details only for the cases $r,s\ne\infty$).

It follows from Lemma 3.2 that

$$N^{\varphi}_{sr}(Sf)\le C\left\{\sum_j\left[\sum_l\left(\sum_k|f(\xi^k_{lj}|2^{j(-2+1/s)}\varphi(2^j)|Q^k_{lj}|\right)^{s}\right]^{r/s}\right\}^{1/r}$$

$$\text{but,} \quad \sum_{k=1}^{M^2} |f(\xi_{lj}^k)| 2^{j(-2+1/s)} \varphi(2^j) |Q_{lj}^k| \le C_M \varphi(2^j) 2^{j/s} \sup_{\xi \in Q_{lj}} |f(\xi)| .$$

It now follows from Lemma 3.3 that $N_{sr}^{\varphi}(Sf) \le C_M N_{sr}^{\varphi}(f)$. Therefore, parts i) and ii) of (4.2) are verified.

Using Proposition 2.3 and Lemma 3.3 we find that $N_{sr}^{\varphi}(f - Sf) \le \frac{C}{M} N_{sr}^{\varphi}(f)$.

The constant C in this last relation does not depend on M, so we may choose M so large that $\frac{C}{M} < 1/2$ and the proof is complete.

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