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$\mathcal{M}$–HARMONIC BLOCH SPACE AND *BMO* IN THE BERGMAN METRIC ON THE UNIT BALL

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Abstract. *In [2] it was shown that many characterizations of analytic Bloch functions also characterize $\mathcal{M}$–harmonic Bloch functions. In this paper we give several additional characterizations of $\mathcal{M}$–harmonic Bloch functions on the unit ball B of C^n.*

1. Introduction. In this paper we continue the investigation of $\mathcal{M}$–harmonic Bloch space began in [2], where it was shown that many characterizations of analytic Bloch functions also characterize $\mathcal{M}$–harmonic Bloch functions.

The main purpose of this paper is to give several additional characterizations of $\mathcal{M}$–harmonic Bloch functions on the unit ball B of C^n.

As in [5], we say that a function $u \in C^2(B)$ is $\mathcal{M}$–harmonic in B, $f \in \mathcal{M}$, if $\tilde{\Delta} u(z) = 0$ for every $z \in B$. The operator $\tilde{\Delta}$ is the invariant Laplacian defined by $\tilde{\Delta} u(z) = \Delta(u \circ \varphi_z)(0)$, $z \in B$, where Δ is the ordinary Laplacian and φ_z the standard automorphizm of B ($\varphi_z \in \operatorname{Aut}(B)$) taking 0 to z (see [5]).

For $f \in C^1(B), Df = \left(\dfrac{\partial f}{\partial z_1}, \dots, \dfrac{\partial f}{\partial z_n}\right)$ denotes the complex gradient of f, $\nabla f = \left(\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_{2n}}\right)$, $z_k = x_{2k-1} + ix_{2k}$, $k = 1, 2, \dots, n$, denotes the real gradient of f.

For $f \in C^1(B)$ let $\tilde{D} f(z) = D(f \circ \varphi_z)(0)$, $z \in B$, and $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$, $z \in B$, be the invariant complex gradient of f and the invariant real gradient of f respectively.

We say that $f \in C^1(B)$ is a Bloch function, $f \in \mathcal{B}$, if $||f||_{\mathcal{B}} = \sup\limits_{z \in B} |\tilde{\nabla} f(z)| < \infty$.

Let $\beta(\cdot, \cdot)$ be a Bergman metric on B. By definition ([3],p.45) β is the "integrated form" of the infinitesimal metric

$$G_z = (g_{ij}(z)) = \frac{1}{2}\left(\frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K(z, z)\right), \text{where} \quad K(z, w) = (1 - \langle z, w\rangle)^{-n-1}$$

is the Bergman reproducing kernel for B.

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Let $\|\cdot\|_\beta$ denote the Lipschitz norm; i.e. if f is a continuous function on B. Then $\|f\|_\beta$ is the smallest $A \geq 0$ for which $|f(z) - f(w)| \leq A\beta(z,w)$, $z, w \in B$.

We say that $f \in \operatorname{Lip}\beta$ if $\|f\|_\beta < \infty$.

For $f \in C^1(B)$, define

$$Q_f(z) = \sup_{w \neq 0} \left\{ \frac{(|\langle Df(z), \bar{w}\rangle|^2 + |\langle D\bar{f}(z), \bar{w}\rangle|^2)^{1/2}}{\sqrt{\langle G_z w, w\rangle}} \right\} \quad \text{and} \quad \|f\|_Q = \sup_{z \in B} Q_f(z).$$

We define Q to be the space of functions $f \in C^1(B)$ such that $\|f\|_Q < \infty$.

For $f \in C(B)$, we define $\operatorname{Osc}(f)(z) = \sup\{\,|f(z) - f(w)| : w \in E(z,1)\,\}$, where $E(z,r) = \{\, w \in B : \beta(z,w) \leq r \,\}$, $0 < r < \infty$.

We say that $f \in C(B)$ is of bounded oscillation, $f \in BO$, if $\|f\|_{\mathrm{Osc}} = \sup_{z \in B} \operatorname{Osc}(f)(z) < \infty$.

Let $|E(z,r)| = \nu(E(z,r))$, where ν is a normalized Lebesgue measure on B.

For fixed $r > 0$ and $f \in L^2(B)$, we define

$$\hat{f}(z,r) = \frac{1}{|E(z,r)|} \int_{E(z,r)} f(w)\, d\nu(w) \quad \text{and}$$

$$MO_r(f)(z) = \left(\frac{1}{|E(z,r)|} \int_{E(z,r)} |f(w) - \hat{f}(z,r)|^2\, d\nu(w) \right)^{1/2}.$$

We say $f \in L^2(B)$ is in BMO_r if $\|f\|_{BMO_r} = \sup_z MO_r(f)(z) < \infty$, and f is in BMO_r^* if

$$\|f\|_{BMO_r^*} = \sup_{z \in B} \left(\frac{1}{|E(z,r)|} \int_{E(z,r)} |f(w) - f(z)|^2\, d\nu(w) \right)^{1/2} < \infty.$$

Given $f \in L^2(B)$, let

$$MO(f)(z) = \left(\frac{1}{2} \int_B \int_B |f(u) - f(v)|^2 |K_z(u)|^2 |K_z(v)|^2\, d\nu(u)\, d\nu(v) \right)^{1/2}, \quad z \in B,$$

where $K_z(u) = K(z,z)^{-1/2} K(u,z)$.

We say $f \in L^2(B)$ is in BMO if $\|f\|_{BMO} = \sup_{z \in B} MO(f)(z) < \infty$.

THEOREM. *For $f \in \mathcal{M} \cap L^2(B)$, the following are equivalent:*

(i) $f \in \mathcal{B}$, (ii) $f \in Q$,

(iii) $f \in \operatorname{Lip}\beta$, (iv) $f \in BO$,

(v) $f \in BMO$, (vi) $f \in BMO_r$ *for all* $r > 0$,

(vii) $f \in BMO_r$ *for some* $r > 0$, (viii) $f \in BMO_r^*$ *for all* $r > 0$,
(ix) $f \in BMO_r^*$ *for some* $r > 0$.

For holomorphic functions the theorem was proved in [6] and [1].

We don't need in this note the characterizations of $\mathcal{M}$−harmonic Bloch functions obtained in [2], but we state them for a reader's convenience.

THEOREM J.P. *Let* $f \in \mathcal{M}$. *Then the following are equivalent:*

(i) $f \in \mathcal{B}$,

(ii) $\sup\limits_{z\in B} \left(\tilde{\Delta}|f|^2\right)^{1/2} < \infty$,

(iii) $\sup\limits_{z\in B} \sqrt{1-|z|^2}\left(|Df(z)|^2 - |Rf(z)|^2 + |D\bar{f}(z)|^2 - |R\bar{f}(z)|^2\right)^{1/2} < \infty$,

(iv) $\sup\limits_{z\in B}(1-|z|^2)|\nabla f(z)| < \infty$,

(v) $\sup\limits_{z\in B}(1-|z|^2)\left(|Rf(z)| + |R\bar{f}(z)|\right) < \infty$,

Here, as usual, R *denotes the radial derivative* $R = \sum_{j=1}^{n} z_j \dfrac{\partial}{\partial z_j}$.

2. Proof of Theorem. To show (i) $\Rightarrow$ (ii) we need the following lemma.

LEMA 2.1. *Let* $f \in C^1(B)$. *Then* $Q_f(\varphi(z)) = Q_{f\circ\varphi}(z)$, $z \in B$, *for all* $\varphi \in$ Aut(B).

PROOF. Notice that each $\varphi \in \mathrm{Aut}(B)$ is an isometry on B in the Bergman metric. This statement is expressed by the following equation:

$$\langle G_{\varphi(z)} J_z(\varphi)w, J_z(\varphi)w\rangle = \langle G_z w, w\rangle \text{ , for all } z \in B\, ;\, w \in C^n\, ;\, \varphi \in \mathrm{Aut}(B)\,.$$

Here $J_z(\varphi)$ denotes the complex Jacobian matrix of φ at z.

Thus, since $J_z(\varphi)$ is an invertible matrix, we have $Q_{f\circ\varphi}(z) =$

$$= \sup_{w\neq 0}\left\{\frac{(|\langle D(f\circ\varphi)(z), \bar{w}\rangle|^2 + |\langle D(\bar{f}\circ\varphi)(z), \bar{w}\rangle|^2)^{1/2}}{\sqrt{\langle G_z w, w\rangle}}\right\}$$

$$= \sup_{w\neq 0}\left\{\frac{(|\langle D(f\circ\varphi)(z), \bar{w}\rangle|^2 + |\langle D(\bar{f}\circ\varphi)(z), \bar{w}\rangle|^2)^{1/2}}{\sqrt{\langle G_{\varphi(z)} J_z(\varphi)w, J_z(\varphi)w\rangle}}\right\}$$

$$= \sup_{w\neq 0}\left\{\frac{\left(\left|\left\langle D(f\circ\varphi)(z), \overline{(J_z(\varphi))^{-1}w}\right\rangle\right|^2 + \left|\left\langle D(\bar{f}\circ\varphi)(z), \overline{(J_z(\varphi))^{-1}w}\right\rangle\right|^2\right)^{1/2}}{\sqrt{\langle G_{\varphi(z)} w, w\rangle}}\right\}$$

$$= \sup_{w\neq 0}\left\{\frac{(|\langle Df(z), \bar{w}\rangle|^2 + |\langle D\bar{f}(z), \bar{w}\rangle|^2)^{1/2}}{\sqrt{\langle G_{\varphi(z)} w, w\rangle}}\right\} = Q_f(\varphi(z)).$$

(i) ⇒ (ii). Since $\alpha^2 = \inf_{|w|=1} \langle G_0 w, w\rangle > 0$, it follows from the definition of $Q_f(z)$ that

$$Q_{f\circ\varphi_z}(0) \le \frac{1}{\alpha}\left(|D(f\circ\varphi_z)(0)|^2 + |D(\bar f\circ\varphi_z)(0)|^2\right)^{1/2} =$$
$$= \frac{1}{\alpha}\left(|\tilde D f(z)|^2 + |\tilde D \bar f(z)|^2\right)^{1/2} = \frac{1}{\alpha\sqrt{2}}|\tilde\nabla f(z)|,$$

and so, by Lemma 2.1,

$$||f||_Q = \sup_{z\in B} Q_f(z) = \sup_{z\in B} Q_f(\varphi_z(0)) = \sup_{z\in B} Q_{f\circ\varphi_z}(0) \le C\sup_{z\in B}|\tilde\nabla f(z)| = C||f||_{\mathcal{B}}.$$

Here and elsewhere constants are denoted by C which may indicate a different constant from one occurrence to the next.

(ii) ⇒(iii) . Fix $z, w \in B$. Let $\gamma : [0,1] \to B$ be a geodesic (in the Bergman metric) with $\gamma(0) = z$ and $\gamma(1) = w$. Then

$$|f(z) - f(w)| = \left|\int_0^1 \frac{d}{dt} f(\gamma(t))\,dt\right| \le \int_0^1 \left|\left\langle Df(\gamma(t)), \overline{\gamma'(t)}\right\rangle + \overline{\left\langle D\bar f(\gamma(t)), \overline{\gamma'(t)}\right\rangle}\right| dt$$
$$\le \int_0^1 \left(\left|\left\langle Df(\gamma(t)), \overline{\gamma'(t)}\right\rangle\right| + \left|\left\langle D\bar f(\gamma(t)), \overline{\gamma'(t)}\right\rangle\right|\right) dt$$
$$\le \sqrt{2}\int_0^1 Q_f(\gamma(t))\sqrt{\langle G_{\gamma(t)}\gamma'(t), \gamma'(t)\rangle}\,dt \le \sqrt{2}||f||_Q\beta(z,w).$$

Thus, $||f||_\beta \le \sqrt{2}||f||_Q$.

(iii) ⇒ (iv) follows from the following characterization of the class BO.

LEMMA 2.2 ([1], p. 329) *For a continuous function f on B, the following are equivalent:*

(1) f *is in* BO,

(2) *there is a constant* $C = C_f > 0$ *with* $|f(z) - f(w)| \le C + C\beta(z,w)$,

for all z, w *in* B,

(3) $\sup_{z\in B} ||f\circ\varphi_z - f(z)||_{L^p(B)} < \infty$ *for all* $p > 0$.

(iv)⇒(v) follows from the identity $||f\circ\varphi_z - f(z)||_{L^2(B)} = MO(f)(z)$ and Lemma 2.2.

(v)⇒(vi). Since $|E(z,r)| \cong |1 - \langle w, z\rangle|^{n+1} \cong (1-|z|^2)^{n+1}$, $w \in E(z,r)$, we have

$$MO(f)(z)^2 \ge \frac{C_r}{|E(z,r)|^2}\int_{E(z,r)}\int_{E(z,r)} |f(u) - f(v)|^2\,d\nu(u)\,d\nu(v) = 2C_r\,MO_r(f)(z)^2$$

The desired result follows at once.

LEMMA 2.3.([1], p. 331). *For any fixed r, BMO_r is contained in BMO. Moreover, there is a constant C_r, such that for all f in BMO_r, $\|f\|_{BMO} \le C_r\|f\|_{BMO_r}$.*

Hence, **(v)⇔(vi)⇔(vii)**.

LEMMA 2.4. *For any fixed r, $\mathcal{M} \cap BMO_r = \mathcal{M} \cap BMO_r^*$.*

PROOF. For $f \in BMO_r^*$, we find, by use of the Cauchy-Schwarz inequality, that

$$MO_r(f)(z) = \left(\frac{1}{2|E(z,r)|^2}\int_{E(z,r)}\int_{E(z,r)} |f(u)-f(v)|^2 d\nu(u)d\nu(v)\right)^{1/2} \le$$

$$\le \sqrt{2}\left(\frac{1}{|E(z,r)|}\int_{E(z,r)} |f(w)-f(z)|^2 d\nu(w)\right)^{1/2}.$$

Thus, $BMO_r^* \subset BMO_r$.

Conversely, let $f \in \mathcal{M} \cap BMO_r$. Then

$$f(z) = \int_B h(\varphi_z(w))f(w)\,d\tau(w), \quad z \in B,$$

where h is a radial function which belongs to $C^\infty(B)$ and with the compact support such that

$$\int_B h(w)\,d\tau(w) = 1 \quad \text{and} \quad d\tau(z) = (1-|z|^2)^{-n-1}\,d\nu(z).$$

Since τ is $\mathcal{M}$−invariant, we have $\int_B h(\varphi_z(w))\,d\tau(w) = 1$, $z \in B$.

Thus,

$$|f(z) - \hat{f}(z,r)| = \left|\int_B h(\varphi_z(w))f(w)\,d\tau(w) - \int_B \hat{f}(z,r)h(\varphi_z(w))\,d\tau(w)\right|$$

$$\le \int_B |h(\varphi_z(w))||f(w)-\hat{f}(z,r)|\,d\tau(w).$$

By a suitable choice of function h we get

$$|f(z)-\hat{f}(z,r)| \le \frac{C}{|E(z,r)|}\int_{E(z,r)} |f(w)-\hat{f}(z,r)|\,d\nu(w).$$

Using this we find that

$$\|f\|_{BMO^*_r} = \sup_{z\in B}\left(\frac{1}{|E(z,r)|}\int\limits_{E(z,r)} |f(w)-f(z)|^2\,d\nu(w)\right)^{1/2}$$

$$\le \|f\|_{BMO_r} + \sup_{z\in B}|f(z)-\hat f(z,r)| \le C\|f\|_{BMO_r}.$$

As a consequence, we have **(vi)**$\Leftrightarrow$**(vii)**$\Leftrightarrow$ **(viii)**$\Leftrightarrow$**(ix)**.

To finish the proof of Theorem it remains to show that **(ix)**$\Rightarrow$**(i)**.

Since f is $\mathcal{M}-$harmonic, by Theorem 2.1 ([4]), we have

$$|\tilde\nabla f(z)|^2 \le C\int\limits_{E(z,r)} |f(w)|^2\,d\tau(w), \quad \text{and hence}$$

$$|\tilde\nabla f(z)|^2 \le C\int\limits_{E(z,r)} |f(w)-f(z)|^2\,d\tau(w) \le \frac{C}{|E(z,r)|}\int\limits_{E(z,r)} |f(w)-f(z)|^2\,d\nu(w).$$

Therefore, $\|f\|_{\mathcal{B}} \le C\|f\|_{BMO^*_r}$.

This finishes the proof of Theorem.

Added in proof. After I had submitted the paper for publication I realized that K. Hahn and E. Yaussfi also had obtained some of the equivalences in (i) through (ix) in the paper:

$\mathcal{M}-$Harmonic Besov $p-$spaces and Hankel Operators in the Bergman Space on the Ball in C^n, Manuscripta Math., 71(1991), 67-81.

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