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AN APPLICATION OF THE LAPLACE AREOLAR INTEGRAL FOR SOLVING AREOLAR LINEAR EQUATIONS

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ABSTRACT. *Areolar linear equations have been investigated in many papers (for example see [2]–[4]). In the first part of this article, we introduce a generalization of the Laplace areolar integral (2). The physical and technical meaning of the areolar Laplace integral is given in [1]. In the second part we search for the solution $W = W(z,\bar{z})$ of an areolar linear homogeneous equation (5) in the form of contour Laplace areolar integral (4).*

I. The mapping

$$L_a : f \to F_a \tag{1}$$

is defined by the equation

$$\int_C e^{\bar{z}\zeta} f(\zeta)\, d\zeta = F_a(z) \tag{2}$$

where $\zeta = \xi + i\eta$ and $z = x + iy$ are independent complex variables, C is a piecewise differentiable curve and $f = f(\zeta)$ a complex function such that integral in (2) exists. L_a is a Laplace areolar transformation and integral in (2) is called Laplace areolar integral (see [1]).

In the second part of this paper we will suppose that C is a closed piecewise differentiable curve, and f an analytic function of ζ in $\operatorname{int} C$ and continuous in $C \cup \operatorname{int} C$ exceipt in finitely many singular points $\zeta_1, \zeta_2, \dots, \zeta_m$ in $\operatorname{int} C$.

A natural generalization of the Laplace areolar transformation L_a (2) is the case when the function f depends on some parameter p independent from ζ, $f = f(p,\zeta)$. In this case, the function F_a will also depend on the parameter p:

$$\int_C e^{\bar{z}\zeta} f(p,\zeta)\, d\zeta = F_a(p,z) \tag{3}$$

THEOREM 1 *If $f = f(p,\zeta)$ is an analytic function of $\zeta = \xi + i\eta$ $\left(\frac{\partial f}{\partial \bar{\zeta}} = 0\right)$ and of parameter p $\left(\frac{\partial f}{\partial \bar{p}} = 0\right)$, then the function F_a, defined by the equation (3), is an antianalytic function of z $\left(\frac{\partial F_a}{\partial z} = 0\right)$ and analytic function of p $\left(\frac{\partial F_a}{\partial \bar{p}} = 0\right)$.*

PROOF. First, note that the symbols $\frac{\partial}{\partial\zeta}$ and $\frac{\partial}{\partial\bar{\zeta}}$, which appear in this theorem are symbols for the operator derivatives in ζ and $\bar{\zeta}$ defined by the equations:

$$\begin{aligned}\frac{\partial f}{\partial\zeta} &= \frac{1}{2}\left[U_\xi + V_\eta + i(V_\xi - U_\eta)\right]\\ \frac{\partial f}{\partial\bar{\zeta}} &= \frac{1}{2}\left[U_\xi - V_\eta + i(V_\xi + U_\eta)\right]\\ f &= f(\zeta) = U(\xi,\eta) + iV(\xi,\eta) \qquad \text{([7] page 17).}\end{aligned}$$

G. V. Kolosov noted that for $\frac{\partial}{\partial\zeta}$ and $\frac{\partial}{\partial\bar{\zeta}}$ all conditions for sums, differences, products and quotients for the ordinary derivatives are correct [5].

By Cauchy theorem on residues

$$F_a(p,z) = \int_C e^{\bar{z}\zeta} f(p,\zeta)\,d\zeta = 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{\zeta_k=\zeta_k(p)} \left(e^{\bar{z}\zeta} f(p,\zeta)\right) \quad .$$

Here p is a parameter independent of ζ, and C piecewise smooth curve. Since p is also a parameter in the function F_a we will put $p = p_0 = const$. So, we have

$$\frac{\partial F_a(p_0,z)}{\partial z} = 2\pi i \sum_{k=1}^{m} \frac{\partial}{\partial z} \operatorname{Res}_{\zeta_k(p_0)} \left(e^{\bar{z}\zeta} f(p_0,\zeta)\right) \quad .$$

The function right $e^{\bar{z}\zeta} f(p_0,\zeta)$ depends on ζ and z. Since z and ζ are independent, we may change the places of $\frac{\partial}{\partial z}$ and Res ([6], page 235). So, we have

$$\begin{aligned}\frac{\partial F_a(p_0,z)}{\partial z} &= 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{\zeta_k(p_0)} \frac{\partial}{\partial z}\left(e^{\bar{z}\zeta} f(p_0,\zeta)\right) =\\ &= 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{\zeta_k(p_0)} \left[f(p_0,\zeta)\frac{\partial}{\partial z}(e^{\bar{z}\zeta}) + e^{\bar{z}\zeta}\frac{\partial}{\partial z}(f(p_0,\zeta))\right] =\\ &= 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{\zeta_k(p_0)} 0 = 0.\end{aligned}$$

Since p_0 is arbitrary, we have $\frac{\partial F_a(p,z)}{\partial z} = 0$, which means that F_a is an antianalytic function of z ([7], page 19).

For the second statement of the theorem we have

$$\frac{\partial F_a(p,z)}{\partial\bar{p}} = \frac{\partial}{\partial\bar{p}}\int_C e^{\bar{z}\zeta} f(p,\zeta)\,d\zeta = \int_C e^{\bar{z}\zeta}\frac{\partial f(p,\zeta)}{\partial\bar{p}}\,d\zeta = \int_C e^{\bar{z}\zeta} 0\,d\zeta = 0$$

which means that F_a is an analytic function of p ([7], page 19).

CONSEQUENCE. *The theorem above says that F_a is a function which " explicitly" depends on p and $\bar{z}$ and so is a function of the form*

$$F_a = F_a(p, z) = W(p, \bar{z}),$$

where W is an analytic function of p and $\bar{z}$.

In agreement with this consequence, we may write the equation (3)

$$\int_C e^{\bar{z}\zeta} f(p, \zeta)\, d\zeta = W(p, \bar{z}) \quad .$$

If in the last equation we put $p = z$, then

$$\int_C e^{\bar{z}\zeta} f(z, \zeta)\, d\zeta = W(z, \bar{z}) \quad . \tag{4}$$

The substitution of p with z we may justify since $z = x + iy$ and $\bar{z} = x - iy$ satisfy

$$\frac{D(z, \bar{z})}{D(x, y)} = \begin{vmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \\ \dfrac{\partial \bar{z}}{\partial x} & \dfrac{\partial \bar{z}}{\partial y} \end{vmatrix} = -2i \neq 0,$$

which means that z and $\bar{z}$ are independent, and from the fact that z is a constant when we need operator derivative at $\bar{z}$.

II. Now, we will try to find a solution of the homogeneous linear equation in the form of the Laplace areolar integral (4):

$$\sum_{j=0}^{n} P_j \frac{\partial^{n-j} W}{\partial \bar{z}^{n-j}} = 0 \quad , \tag{5}$$

where $P_j = P_j(z), j = 1, 2, \ldots, n$ are given analytic functions of $z = x + iy$ and $P_0 = 1$.

If we put $W_1 = W$ in the areolar equation (5) then we have a homogeneous system of n linear areolar equations

$$\begin{aligned} \frac{\partial W_k}{\partial \bar{z}} &= W_{k+1} \qquad (k = 1, 2, \ldots, n-1) \\ \frac{\partial W_n}{\partial \bar{z}} &= -\sum_{j=1}^{n} P_{n-j+1} W_j \end{aligned} \tag{6}$$

which is a special case of the general system

$$\frac{\partial W_k}{\partial \bar{z}} = \sum_{j=1}^{n} a_{kj} W_j \qquad (k = 1, 2, \ldots, n), \tag{7}$$

where $a_{kj} = a_{kj}(z)$, $(k, j = 1, 2, \ldots, n)$ are given analytic functions of z.

We try to find a solution of system (7) in the form

$$W_k = \int_C e^{\bar{z}\zeta} f_k(z, \zeta)\, d\zeta \qquad (k = 1, 2, \ldots, n) \tag{8}$$

where $f_k = f_k(z, \zeta)$ are analytic functions in z and ζ $\left(\frac{\partial f_k}{\partial \bar{z}} = 0, \frac{\partial f_k}{\partial \bar{\zeta}} = 0\right)$ which we try to find. In (8) C is a closed piecewise smooth curve and its interior contains all singularities of the functions f_k related to ζ and they are within a finite distance from the coordinate origin.

Taking areolar derivatives

$$\frac{\partial W_k}{\partial \bar{z}} = \int_C \zeta e^{\bar{z}\zeta} f_k(z, \zeta)\, d\zeta \tag{9}$$

of the functions W_k, defined by equations (8), and replacing W_k and $\frac{\partial W_k}{\partial \bar{z}}$ in the system (7) with their own values (8) and (9), after some prearrangements we get n identities in ζ

$$\int_C \left[\sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} f_j(z, \zeta) + (a_{kk} - \zeta) f_k(z, \zeta) \right] e^{\bar{z}\zeta}\, d\zeta \equiv 0 \tag{10}$$

$(k = 1, 2, \ldots, n)$ for determination of the functions $f_k = f_k(z, \zeta)$. This identities will be satisfied if the expressions in the middle brackets are constants with respect to ζ. In that case the functions under integral (10) will be regular functions in the closed domain $C \cup \operatorname{int} C$ of ζ-plane and by Cauchy theorem all these integrals on C will be equal to zero. Since z is an independent value of ζ and $f_k = f_k(z, \zeta)$, it is enough to replace the expressions in the middle brackets in (10) with some analytic functions in z. So to determine the functions $f_k = f_k(z, \zeta)$ we have an algebraic system of n linear equations

$$\sum_{j=1, j \neq k}^{n} a_{kj} f_j(z, \zeta) + (a_{kk} - f_k(z, \zeta)) = -\frac{C_k(z)}{2\pi i} \qquad (k = 1, 2, \ldots, n), \tag{11}$$

where $C_k = C_k(z)$ are some analytic functions in z. By Cramer's formulas, we get

$$f_k(z, \zeta) = \frac{\Delta_k(z, \zeta)}{\Delta(z, \zeta)} \qquad (k = 1, 2, \ldots, n), \tag{12}$$

where

$$\Delta(z,\zeta) = \begin{vmatrix} a_{11}-\zeta & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\zeta & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\zeta \end{vmatrix} = (-1)^n\zeta^n + \dots$$

is the determinant of the system (11) and

$$\Delta_k(z,\zeta) = (-1)^n \frac{C_k(z)}{2\pi i}\zeta^{n-1} + \dots$$

is the determinant of the unknown function $f_k(z,\zeta)$. Putting the functions $f_k(z,\zeta)$ in (8) with their values, determined by formulas (12), we get a solution

$$W_k = \int_C e^{\bar{z}\zeta} \frac{\Delta_k(z,\zeta)}{\Delta(z,\zeta)}\, d\zeta \qquad (k = 1,2,\dots,n) \tag{13}$$

of the system of areolar equations (7) in the form of Laplace areolar integrals.

With the help of Cauchy calculus of residues, it is easy to show that the solution of (13) of the areolar system of equations (7) satisfies the conditions

$$\text{for} \quad \bar{z} = 0, \quad \alpha_0 W_k(z,\bar{z}) = C_k(z) \qquad (k = 1,2,\dots,n) \tag{14}$$

The mark $\alpha_0\omega(z,\bar{z}) = \omega(z,0) = \Phi(z)$, for the boundary conditions for areolar equations, where for $\bar{z}$ we take zero and z is unchanged, is introduced in ([4] page 41–42).

If we choose in system (7)

$$\begin{aligned} a_{k\,k+1} &= 1 && (k = 1,2,\dots,n-1) \\ a_{nj} &= -P_{n-j+1} && (j = 1,2,\dots,n) \\ a_{kj} &= 0, && \text{otherwise} \end{aligned} \tag{15}$$

given system becomes a special system. We correspond this system to the areolar linear equation of order n (5), and the following theorem is true:

THEOREM 2 *Areolar homogeneous linear equation of order* n (5), *with analytic in* z *coefficients* $P_j = P_j(z)$, $(j = 1,2,\dots,n)$ *has a solution in the form of areolar contour Laplace integral*

$$W = W_1 = \int_C e^{\bar{z}\zeta} \frac{\Delta_1(z,\zeta)}{\Delta(z,\zeta)}\, d\zeta$$

which satisfies the boundary conditions

$$\alpha_0 \frac{\partial^k W}{\partial \bar{z}^k} = C_{k+1}(z) \qquad (k = 0,1,2,\dots,n-1)$$

References

[1] D. Dimitrovski, M. Rajović, *Za fizičkata smisla i tehničkoto značenje na areolarnite Laplasovi transformacii* Mat. bilten, 11-12, XXXII-XXXIII, Skopje, (1987–1988), str. 13.

[2] D. Dimitrovski, B. Ilijevski, *L'équation différentielle aréolaire de Laplace et la possibilité de l'introduction des transformations de Laplace aréolaires*, Punime matematike, Priština, 1 (1986), 17–25.

[3] J. Kečkić, *O jednoj klasi parcijalnih jednačina*, Mat. Vesnik, 6(21) (1969), 71–73.

[4] M. Čanak, *Metode diferencijalnih i funkcionalnih jednačina za rešavanje nekih tipova konturnih problema*, Disertacija, Beograd 1976.

[5] Г. В. Колосов, *Об одном приложении теории функций комплексного переменного к плоской задаче математической теории упругости*,Юрьев, 1909 .

[6] В. И. Смирнов, *Курс высшей математики*, Том III, часть вторая, "Наука" - Москва, 1969.

[7] Г. Н. Полжий, *Обобщение теории аналитических функций комплексного переменного*, Киев, 1965.

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