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BOUNDARY SINGULARITIES OF NORMAL SET MAPPINGS UNDER GENERAL BOUNDARY APPROACH

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Abstract. *Some types of singularities generated by cluster sets of normal set–mappings for general boundary approach are investigated. Lindelöf and Meier types theorems are proved.*

1. Preliminaries

Let D denote the unit disc $|z| < 1$ in the complex z–plane, Γ be the circumference $|z| = 1$ and let $d\rho(z) = (1-|z|^2)^{-1}|dz|$ be the linear element of hyperbolic metric in D. We define the notion of σ–porous set introduced by E. P. Dolßenco [2]. For a set E on Γ, a point $\zeta = e^{i\theta}$ of Γ and a real $\epsilon > 0$, we denote by $r(\zeta, E, \epsilon)$ the length of the largest open arc which belongs to the arc $\gamma_{\zeta,\epsilon} = \{\, \xi = e^{i\varphi} : |\varphi - \theta| < \epsilon \,\}$ and does not intersect E (if there is no such an arc, we put $r(\zeta, E, \epsilon) = 0$). The point $\zeta = e^{i\theta}$ is called a point of porousity of the set E if

$$r(\zeta, E) = \limsup_{\epsilon \to 0} \frac{r(\zeta, E, \epsilon)}{\epsilon} > 0 \quad .$$

The set E is called a porous set on Γ if every point of E is a point of porousity for E. A set on Γ is called a σ- porous set if it is the union of not more than a countable collection of porous sets.

It follows from the definition, that any porous set is nowhere dense on Γ, and therefore, any σ-porous set is a set of first Baire category on Γ. Moreover, by the definition, no point of porousity of a measurable set E on Γ can be the Lebesgue density point of E. Since, by the Lebesgue theorem, almost all points of a measurable set are its density points, it follows, that any porous set and any σ-porous set is of linear Lebesque measure zero on Γ. The converse assertions are not, in general, true (for a discussion on the subject, see, for instance, the papers of N. Yanagihara [8] and D. C. Rung [5]).

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2. The geometry of boundary paths in the unit disc

In this paragraph we consider the geometry of boundary paths in D for a general approach function. Following the paper of D. C. Rung [5], a real nonnegative continuous function $h(x)$ defined on a segment $I_h = [-l, l], 0 < l \leq \pi$, of the real axes is called an approach function if $h(x)$ is the even function on I_h which is strictly increasing for $x \geq 0$ and $h(0) = 0, h(\pm l) = 1$. We put $h_a(x) = \min\left\{\dfrac{h(x)}{a}, 1\right\}$ for an arbitrary $a > 0$.

For a fixed point $\zeta = e^{i\theta}$ on Γ and for arbitrary $b > a > 0$, we consider in D a boundary h-curve

$$L_h(\zeta, a) = \{\, [1 - h_a(\varphi - \theta)]e^{i\varphi} : (\varphi - \theta) \in I_h \,\},$$

which passes through the point $z = 0$ and touches Γ at the point ζ, and two sets

$$R\Delta_h(\zeta, a, b) = \{re^{i\varphi} \in D : 1 - h_a(\varphi - \theta) < r < 1 - h_b(\varphi - \theta); \theta \leq \varphi \leq \theta + l\}$$

$$L\Delta_h(\zeta, a, b) = \{re^{i\varphi} \in D : 1 - h_a(\varphi - \theta) < r < 1 - h_b(\varphi - \theta); -l + \theta \leq \varphi \leq \theta\},$$

which are called a right h-angle and a left h-angle with the vertex at the point ζ, respectively. For the sake of simplicity, we use the symbol $\Delta_h(\zeta, a, b)$, or simply, $\Delta_h(\zeta)$ for a right or a left h-angle, and the symbol $L_h(\zeta)$ for an arbitrary h-curve at ζ. We note that an h-angle $\Delta_h(\zeta, a, b)$ is a subset of D which is contained between the h-curves $L_h(\zeta, a)$ and $L_h(\zeta, b)$.

We denote $\rho(z; L_h(\zeta, a)) = \inf\{\,\rho(z, w) ; w \in L_h(\zeta, a)\,\}$ for a point $z \in D$ and for an h-curve $L_h(\zeta, a)$, where $\rho(z, w)$ denotes the hyperbolic distance between the points z and w in D.

LEMMA 1. (U. U. Stayanarayana, W. L. Weiss [6]) *If an approach function $h(x)$ is convex down on I_h, then for any $L_h(\zeta, a)$ and $L_h(\zeta, b)$*

$$(1) \qquad \lim_{\substack{z \to \zeta \\ z \in L_h(\zeta, a)}} \rho(z; L_h(\zeta, b)) = \lim_{\substack{z \to \zeta \\ z \in L_h(\zeta, b)}} \rho(z; L_h(\zeta, a)) = \frac{1}{2} \log \frac{|a + b| + |a - b|}{|a + b| - |a - b|} \;.$$

3. Cluster sets and boundary singularities of arbitrary set-mappings

Let X and Y be arbitrary topological spaces. For an element p in X, we denote by $\mathcal{U}_p$ the system of all neighborhoods of the point p in X, and for a set A in X (or A in Y), the symbol $\overline{A}$ stands for the closure of the set A in X (or in Y). We denote by $P(Y)$ the collection of all subsets of Y including the empty set $\emptyset$.

Consider a non-empty set A in X and a set-mapping $S : A \to P(Y)$. For any subset A_I of A, we put $S(A_I) = \cup_{p \in A_I} S(p); S(\emptyset) = \emptyset$. Let Q be a subset of A and let a point p belongs to the closure $\overline{Q}$. The cluster set $C(S, p, Q)$ of the set-mapping S at the point p along the set Q is defined as the intersection

$$C(S, p, Q) = \bigcap \overline{S(U \cap Q)}$$

taken over all neighborhoods U in $\mathcal{U}_p$.

Let now X be the complex z-plane C, A be the unit disc D : $|z| < 1$ in C and let Γ : $|z| = 1$. A topological space Y is assumed to be compact. We introduce some sets of boundary singularities for an arbitrary set-mapping $S : D \to P(Y)$. Denote by $K_h(S)$ the set of all points ζ on Γ at which $C\left(S,\zeta,\Delta_h^1(\zeta)\right) = C\left(S,\zeta,\Delta_h^2(\zeta)\right)$ for any two h-angles $\Delta_h^1(\zeta)$ and $\Delta_h^2(\zeta)$. A point ζ of $K_h(S)$ belongs to the set $C_h(S)$ if $C(S,\zeta,\Delta_h(\zeta)) = C(S,\zeta,D)$, and a point ζ of $C_h(S)$ belongs to the set $I_h(S)$ if $C(S,\zeta,D) = Y$.

A point ζ on Γ is said to belong to the set $L_h(S)$ if $C\left(S,\zeta,L_h^1(\zeta)\right) = C\left(S,\zeta,L_h^2(\zeta)\right) \neq Y$ for any two h-curves $L_h^1(\zeta)$ and $L_h^2(\zeta)$. A point ζ of $L_h(S)$ belongs to the set $M_h(S)$ if $C\left(S,\zeta,L_h(S)\right) = C(S,\zeta,D) \neq Y$, and a point ζ on Γ belongs to the set $I_h^+(S)$ if $C\left(S,\zeta,L_h(\zeta)\right) = Y$ for any h-curve $L_h(\zeta)$. At a point ζ of $I_h^+(S)$, we have $C\left(S,\zeta,L_h(\zeta)\right) = Y = C\left(S,\zeta,\Delta_h(\zeta)\right) = C(S,\zeta,D)$ for any h-curve $L_h(\zeta)$ and any h-angle $\Delta_h(\zeta)$, and therefore, $I_h^+(S) \subset I_h(S)$.

We note also that $I_h^+(S) \cap M_h(S) = I_h^+(S) \cap L_h(S) = \emptyset$.

THEOREM 1. (Ž. S. Oganesjan [4]) *Let Y be a compact metric space. If an approach function $h(x)$ is convex down on I_h , then for an arbitrary set-mapping $S : D \to P(Y)$ the set $\Gamma \setminus C_h(S)$ is an F_σ-set of first Baire category on Γ.*

THEOREM 2. (Ž. S. Oganesjan [4]) *Let Y be a compact metric space. Let an approach function $h(x)$ be convex down on I_h and its inverse function $\mu(t) = h^{-1}(x)$ satisfies the condition*

$$\liminf_{t\to 0} \frac{\mu'(\alpha t)}{\mu'(t)} > 0 \tag{2}$$

for any $\alpha > 1$, where $\mu'(t)$ denotes the derivative of $\mu(t)$ at any point t at which the derivative exists. Then for an arbitrary set-mapping $S : D \to P(Y)$ the set $\Gamma \setminus K_h(S)$ is a σ-porous set of type $G_{\delta\sigma}$ on Γ.

4. Cluster sets of normal set-mappings

Let $Y = (Y,d)$ be a metric space with the metric d. For an element y of Y and a set Q in Y, we denote, as usually, $d(y;Q) = \inf\{\, d(y,y') : y' \in Q \,\}$. Let $\epsilon > 0$ be given and $\mathcal{N}(Q;\epsilon)$ denote the ϵ-neighborhood of the set Q, that is, $\mathcal{N}(Q;\epsilon) = \{\, y \in Y : d(y;Q) < \epsilon \,\}$.

DEFINITION. (S. Yamashita [7]) *A set-mapping $S : D \to P(Y)$ is said to be normal if for any $\epsilon > 0$ there exists a number $\delta = \delta(\epsilon), \delta > 0$ such that for any points z_1 and z_2 in D satisfying the condition $\rho(z_1,z_2) < \delta$ the inclusions $\mathcal{N}\left(S(z_1);\epsilon\right) \supset S(z_2)$ and $\mathcal{N}\left(S(z_2);\epsilon\right) \supset S(z_1)$ hold.*

THEOREM 3. *Let $Y = (Y,d)$ be a compact metric space. Let an approach function $h(x)$ be a convex down on I_h. Then for any normal set-mapping $S : D \to P(Y)$ and any h-curve $L_h(\zeta,a)$, the assertion*

$$C\left(S,\zeta,L_h(\zeta,a)\right) = \cap C\left(S,\zeta,\Delta_h(\zeta)\right) \tag{3}$$

holds, where the intersection is taken over all h-angles $\Delta_h(\zeta)$ containing the h-curve $L_h(\zeta, a)$.

PROOF. The inclusion

$$C(S, \zeta, L_h(\zeta, a)) \subset \cap C(S, \zeta, \Delta_h(\zeta)) \tag{4}$$

is true for an arbitrary set-mapping $S : D \to P(Y)$ by the definition of cluster sets in it.

We need to prove the converse inclusion

$$C(S, \zeta, L_h(\zeta, a)) \supset \cap C(S, \zeta, \Delta_h(\zeta)). \tag{5}$$

Suppose, that (5) is not true. Then, there exists an h-curve $L_h(\zeta, a), a > 0$, such that the inclusion

$$C(S, \zeta, L_h(\zeta, a)) \subset \cap_{m=1}^{\infty} C(S, \zeta, \Delta_h(\zeta, a - 1/m, a + 1/m)) \tag{6}$$

is strict.

Hence, there exists an element y in Y such that y belongs to $C(S, \zeta, \Delta_h(\zeta, a - 1/m, a + 1/m))$ for any $m \in N$, and y does not belong to $C(S, \zeta, L_h(\zeta, a))$. Since the set $C(S, \zeta, L_h(\zeta, a))$ is closed, then by (6), there exists an open ball $B(y, r) = \{y' \in Y : d(y', y) < r\}$, $r > 0$, in Y with the compact closure $\overline{B}(y, r)$ and such that

$$\overline{B}(y, r) \cap C(S, \zeta, L_h(\zeta, a)) = \emptyset \quad . \tag{7}$$

By the choice of the element y, each angle $\Delta_h(\zeta, a - 1/m, a + 1/m), m \in N$, contains a sequence of points $\{z_n^{(m)}\}$ such that $\lim_{n\to\infty} z_n^{(m)} = \zeta$ and $\lim_{n\to\infty} S(z_n^{(m)})(n) = y$, where $S(z_n^{(m)})(n) = y_n^{(m)}$ denotes an element of the set $S(z_n^{(m)})$ in Y.

For fixed $m, n \in N$, we denote by $\tilde{z}_n^{(m)}$ the point on $L_h(\zeta, a)$ at which $\rho(z_n^{(m)}, \tilde{z}_n^{(m)}) = \rho(z_n^{(m)}; L_h(\zeta, a))$. By Lemma 1, for a fixed $m \in N$, we have

$$\lim_{n\to\infty} \rho(z^{(m)}, \tilde{z}^{(m)}) = \frac{1}{2} \log \frac{a + 1/m}{a - 1/m} \quad , \quad m \in N \quad . \tag{8}$$

If we consider the diagonal sequence $\{z_k^{(k)}\}$ and $\{\tilde{z}_k^{(k)}\}$, we get from (8) that

$$\lim_{k\to\infty} \rho(z_k^{(k)}, \tilde{z}_k^{(k)}) = 0 \quad . \tag{9}$$

The sequence $\{z_k^{(k)}\}$ tends to point ζ and the corresponding sequence $\{y_k^{(k)}\}$, $y_k^{(k)} = S(z_k^{(k)})(k), k \in N$, has $\lim_{k\to\infty} y_k^{(k)} = y$. Since the set-mapping S is normal, it follows from (9), that $\lim_{k\to\infty} d\left(S(z_k^{(k)}; S(\tilde{z}_k^{(k)})\right) = 0$. If we denote $S(\tilde{z}_k^{(k)})(k) =$

$\tilde{y}_k^{(k)}, k \in N$, we obtain that $\lim_{k\to\infty} \tilde{y}_k^{(k)} = y$. Since $\tilde{z}_k^{(k)} \in L_h(\zeta, a), k \in N$, we conclude that $y \in C(S, \zeta, L_h(\zeta, a))$. The latter contradicts to (7), and, hence, the inclusion (5) is proved.

Combining (4) and (5), we get (3), and Theorem 3 is proved.

COROLLARY 1. *Let $Y = (Y, d)$ be a compact metric space , and let an approach function $h(x)$ be convex down on I_h. Then for any normal set-mapping $S : D \to P(Y)$ the assertion $C(S, \zeta, L_h(\zeta)) = C(S, \zeta, \Delta_h(\zeta))$ holds at any point ζ of the set $K_h(S)$ for any h-curve $L_h(\zeta)$ and any h-angle $\Delta_h(\zeta)$. In particular, $I_h(S) = I_h^+(S)$.*

5. The Meier and Lindelöf type theorems for normal set-mappings

THEOREM 4. (The Meier type theorem) *Let Y be a compact metric space. If an approach function $h(x)$ is convex down on I_h, then for the arbitrary normal set-mapping $S : D \to P(Y)$ the following assertions holds:*

$$\text{(i)} \qquad C_h(S) = M_h(S) \cup I_h^+(S) \qquad \textit{and} \qquad \text{(ii)} \qquad \Gamma = M_h(S) \cup I_h^+(S) \cup E\,,$$

where E is an F_σ set of first Baire category on Γ.

THEOREM 5. (The Lindelöf type theorem) *Let Y be a compact metric space, and let a set-mapping $S : D \to P(Y)$ be normal. If an approach function $h(x)$ is convex down on I_h, then the following assertion holds:*

$$\text{(i)} \qquad K_h(S) = L_h(S) \cup I_h^+(S)\,.$$

If, in addition, the approach function $h(x)$ satisfies the condition (2) in Theorem 2, then (ii) $\Gamma = L_h(S) \cup I_h^+(S) \cup E$, where E is a σ-porous set of type $G_{\delta\sigma}$ on Γ.

Proof of Theorem 4. To show the validity of the assertion (i), we must prove only the inclusion $C_h(S) \subset M_h(S) \cup I_h^+(S)$, since the inverse inclusion is valid by the definition of the sets. Consider a point ζ of $C_h(S)$. By Corollary 1, $C(S, \zeta, L_h(\zeta, a)) = C(S, \zeta, D)$ for any h-curve $L_h(\zeta, a)$. So, if $C(S, \zeta, D) = Y$, then ζ belongs to $I_h^+(S)$, and if $C(S, \zeta, D) \neq Y$, then ζ belongs to $M_h(S)$.

The assertion (ii) follows from (i) and Theorem 1.

Proof of Theorem 5. Consider a point ζ of the set $K_h(S)$. By Corollary 1, $C(S, \zeta, L_h(\zeta)) = C(S, \zeta, \Delta_h(\zeta))$ for any h-curve $L_h(\zeta)$ and any h-angle $\Delta_h(\zeta)$. If $C(S, \zeta, \Delta_h(\zeta)) \neq Y$, then ζ belongs to $L_h(S)$. If $C(S, \zeta, \Delta(\zeta)) = Y$, then ζ belongs to $I_h(S) = I_h^+(S)$. This proves the inclusion $K_h(S) \subset L_h(S) \cup I_h^+(S)$.

To prove the converse inclusion, we use arguments analogous to those in the proof of Theorem 3. It was noted above, that $I_h^+(S) = I_h(S) \subset K_h(S)$. Consider a point ζ of the set $L_h(S)$ and suppose there exists an h-angle $\Delta_h(\zeta)$ such that $C(S, \zeta, \Delta_h(\zeta)) \neq C(S, \zeta, L_h(\zeta))$ for any h-curve $L_h(\zeta)$. Then, we can choose an element y in $C(S, \zeta, \Delta_h(\zeta))$ which does not belong to $C(S, \zeta, L_h(\zeta))$. Denote by $\{z_n\}$ a sequence of points in $\Delta_h(\zeta)$ for which $\lim_{n\to\infty} z_n = \zeta$, and $\lim_{n\to\infty} y_n = y$, where

$y_n = S(z_n)(n), n \in N$ (cf, the proof of Theorem 3). The sequence $\{z_n\}$ contains a subsequence $\{z_{n_k}\}$ such that $\lim_{k\to\infty} \rho(z_{n_k}; L_h(\zeta, a)) = 0$ for some h-curve $L_h(\zeta, a)$ contained in $\Delta_h(\zeta)$). The contradiction to the assumption follows now in the same way as in the proof of Theorem 3.

REMARK 1. We note that the assertions of Theorems 4 and 5 remain valid for a locally compact metric space Y.

REMARK 2. Theorem 4 is a generalization to set-mappings of improved version of Meier's theorem for meromorphic functions which is obtained by V. I. Gavrilov and A. N. Kanatnikov [2]. In the special case $h(x) = x$, Theorem 4 improves a result of S. Yamashita [7]. Theorem 5 is a generalization to set-mappings of the Lindelöf type theorem for meromorphic functions which is proved by Abdu Al'Rahman Hassan and V. I. Gavrilov [1].

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