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ON THE GENERALIZED PRINCIPAL VALUE OF IMPROPER INTEGRALS

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ABSTRACT. *Majority of the improper integrals we meet in practice are principal values in the sense of Cauchy. However, discontinuous proceses, especially noncomutative and nonasociative processes bring the idea of iterative boundary values* $\lim_{\epsilon_i \to 0}(\lim_{\epsilon_j \to 0} f)$. *In this paper we give a definition of the generalized principal value of the improper real integral, and its calculation.*

1. Introduction. The improper integrals, usually met in practice are *the principal values.* For example:

1° Laplace transformation $\int\limits_0^{+\infty} e^{-zx} f(x)\,dx$;

2° Integral of the construction of the plane wing $\int\limits_0^{\pi/2} \frac{d\phi}{a-\cos\phi}$, $(0 < a < 1)$;

3° Integral of the free fall in physics $\int\limits_0^{h} \frac{dx}{\sqrt{2gx}}$;

are in the sense of the principal value, which is defined by

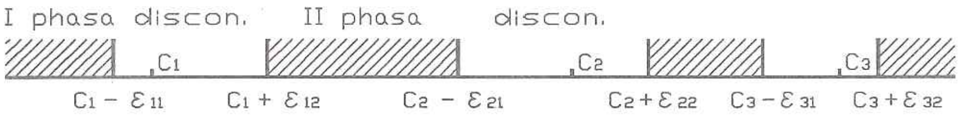
$$v.p. \int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{R \to +\infty \\ \epsilon \to +0}} \left(\int_{-R}^{C_1-\epsilon} f(x)dx + \sum_{i=1}^{n-1} \int_{C_i+\epsilon}^{C_{i+1}-\epsilon} f(x)dx + \int_{C_n+\epsilon}^{+R} f(x)dx \right),$$

where C_i are breaks of $f(x)$, intervals $(C_i - \epsilon, C_i + \epsilon)$ have the same ϵ, and the boundary values are in the sense of "$\epsilon - \delta$" definition.

However, many natural and technical processes and states are not appropriate to this idealistic mathematical model. For example, in chemistry if A, B and C are mutual affine chemical elements, then $(A+B)+C \neq A+(B+C)$. For some elements, the product of the reaction $A+B$ with C, is different from the product of the reaction of A with $B+C$. Also, $A \overrightarrow{+} B \neq B \overrightarrow{+} A$, where in the reaction $A \overrightarrow{+} B$ the arrow means a non-energetic influence of A and B. So, the ordinary associativity and commutativity which are fulfiled for all $\epsilon_i \to 0$ in "$\epsilon - \delta$" definition of the boundary value $\lim\limits_{\epsilon_i \to 0} f(\epsilon_i)$ are not always fulfiled in practice, especially for discontinuous processes. These situations are properly described with the iterative boundary values ,

$$\lim_{\epsilon_1 \to 0} \big(\lim_{\epsilon_2 \to 0} \ldots \big(\lim_{\epsilon_n \to 0} f(\epsilon_i) \big) \ldots \big) ,$$

with a possible dependence between ϵ-s $\epsilon_{12} = \phi(\epsilon_{21})$, $\epsilon_{22} = \psi(\epsilon_{31}) \ldots$



Thus, the value of the improper integral in transition across the disconnection C_i can be multivalued, which depends on the fact how rapidly ϵ_i tend to zero and of dependence between ϵ_i.

Hence, it's necessary to consider the case when the improper integral

$$\lim_{\epsilon_1\to 0}\Big(\lim_{\epsilon_2\to 0}\dots\Big(\lim_{\epsilon_n\to 0} I(\epsilon_1,\epsilon_2,\dots,\epsilon_n)\Big)\dots\Big),$$

has some dependdences between ϵ_i:

$$\epsilon_i = \Phi_i(\epsilon_j)$$

($\epsilon_i = \epsilon_j$ gives the usual principal value in the sense of Cauchy).

For the illustration we give the following examples:

1°. Indefinite integral $\int \frac{dx}{x} = \ln x + C$;

2°. If $0 < a < b$ the integral $\int\limits_a^b \frac{dx}{x} = \ln\frac{b}{a}$;

3°. If $a < 0 < b$ the improper integral

$$v.\int\limits_a^b \frac{dx}{x} \overset{def}{=} \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\left\{\int\limits_a^{-\epsilon_1}\frac{dx}{x} + \int\limits_{\epsilon_2}^b \frac{dx}{x}\right\} = \ln\left|\frac{b}{a}\right| + \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \ln\left|\frac{\epsilon_1}{\epsilon_2}\right|;$$

4°. However, for $\epsilon_1 = \epsilon_2 = \epsilon$ the Cauchy principal value exists

$$v.p.\int\limits_a^b \frac{dx}{x} = \ln\left|\frac{b}{a}\right| + \lim_{\epsilon\to 0}\ln\left|\frac{\epsilon}{\epsilon}\right| = \ln\left|\frac{b}{a}\right|.$$

5°. If $\epsilon_1 = K\epsilon_2$, then we have another value of the integral

$$v.g.\int\limits_a^b \frac{dx}{x} = \ln\left|\frac{b}{a}\right| + \ln K;$$

6°. If $a = R_1 \to -\infty$, $b = R_2 \to +\infty$, then obviously exists

$$v.g.\int\limits_{-\infty}^{+\infty} \frac{dx}{x} = \lim_{\substack{\epsilon_1,\epsilon_2\to 0\\ R_1,R_2\to\infty}} \ln\left|\frac{\epsilon_1 R_2}{\epsilon_2 R_1}\right|,$$

which depend on the possible connections between $\epsilon_1, \epsilon_2, R_1$ and R_2.

The last integrals 5° and 6° we call *the generalized principal values* ($v.g.\int = {}^*\int$).

For $v.g\int$ it is important:

I. We cannot use "$\epsilon - \delta$" definition for lim

II. The order is important in $\lim\limits_{\epsilon_i\to 0}\big(\lim\limits_{\epsilon_j\to 0}\dots\big)$

III. The connections between $\epsilon_i : \epsilon_i = \Phi(\epsilon_j)$

IV. It is possible to appropriate a posteriori, after the part of the process is finished.

In this paper we study the generalization of the improper integral by the theory of complex functions and calculus of residues.

DEFINITION. *Let $f(x)$ be a real function on $[a,b]$, with the poles $C_1, C_2, \dots, C_n$, $a < C_1 < C_2 < \dots < C_n < b$, and let*

$$[a, C_1 - \epsilon_{11}], [C_1 + \epsilon_{12}, C_2 - \epsilon_{21}], [C_2 + \epsilon_{22}, C_3 - \epsilon_{31}], \dots,$$

$$[C_k+\epsilon_{k2}, C_{k+1}-\epsilon_{k+1,1}], \dots, [C_n+\epsilon_{n2}, b]$$

be intervals (phases of the processes), *with* $f(x)$ *continuous. Let* ϵ_{ik} *be connected in the following way:*

$$\text{(L)}\qquad \epsilon_{i1}=\Phi_i(\epsilon_{i2}), \qquad i\le n,$$

$$\text{(P)}\qquad \epsilon_{i+1,1}=\Psi_i(\epsilon_{i2}), \qquad i<n,$$

$$\text{(Q)}\qquad \epsilon_{jk}=\Psi_k(\epsilon_{ik}), \qquad i,j,\le n,\ k=1,2.$$

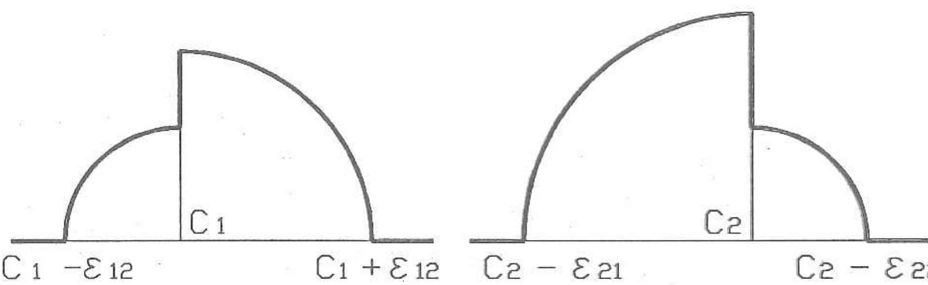
Then the generalized principal value *of the improper integral is*

$$v.g.\int_a^b f(x)\,dx = {}_{(L,P,Q)}{}^{*}\int_a^b f(x)\,dx =$$

$$\lim_{\epsilon_{ik}\to 0}\left(\lim\dots\lim_{\epsilon_{jk}\to 0}\dots\left[\int_a^{C_1-\epsilon_{11}} f(x)\,dx+\sum_{i=1}^{n-1}\int_{C_i+\epsilon_{i2}}^{C_{i+1}-\epsilon_{i+1,1}} f(x)\,dx+\int_{C_n+\epsilon_{n2}}^{b} f(x)\,dx\right]\right).$$

Obviously, the integral $v.g.\int = {}^*\int$ is multivalued, and it can be made simpler in some cases, for which we need a new definition.

Method of complex integration. If C_i is a singularity on a finite distance, then we propose puting the following contour around the singularity



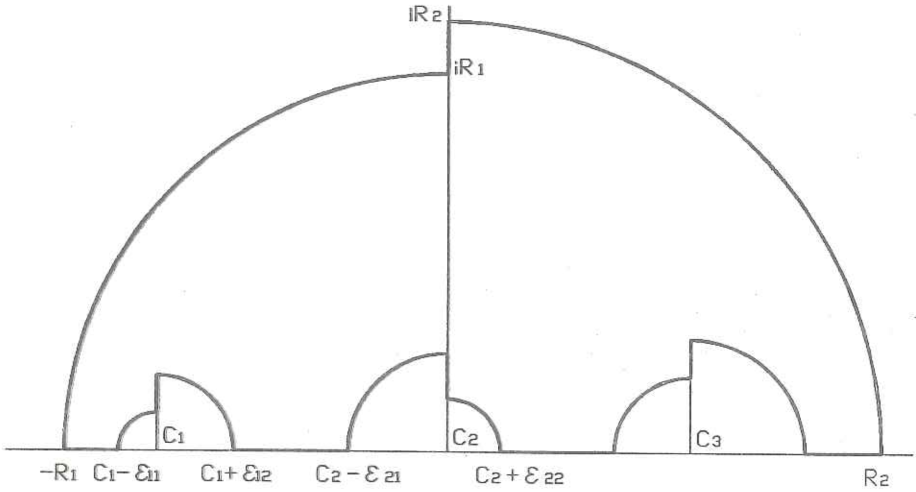
This way we obtain:

I. On the areas quadrants of the circle we used standard Laurent treatment and estimation;

II. Different $\epsilon_{i1}\ne\epsilon_{i2}$ whose generalized principal values are preserved;

III. Integration over x-axis is switched to integration over y-axis and the integral $\int_{C-\epsilon_{i1}}^{C+\epsilon_{i2}} f(x)\,dx$ switches into integral $\int_{\epsilon_{i1}}^{\epsilon_{i2}} f(iy)\,idy$. In a lot of classes of functions this integral is transformed to $\int e^{-y}f(y)\,dy$, which is very easy to calculate and estimate.

The corresponding contours for complex integration for the integral of type $\int_{-\infty}^{+\infty}$ (also for $\int_0^{\infty}$) and $\int_a^b f(x)\,dx$ (including the case when a and b are singularities) are respectivelly


iR2
iR1
C1
C2
C3
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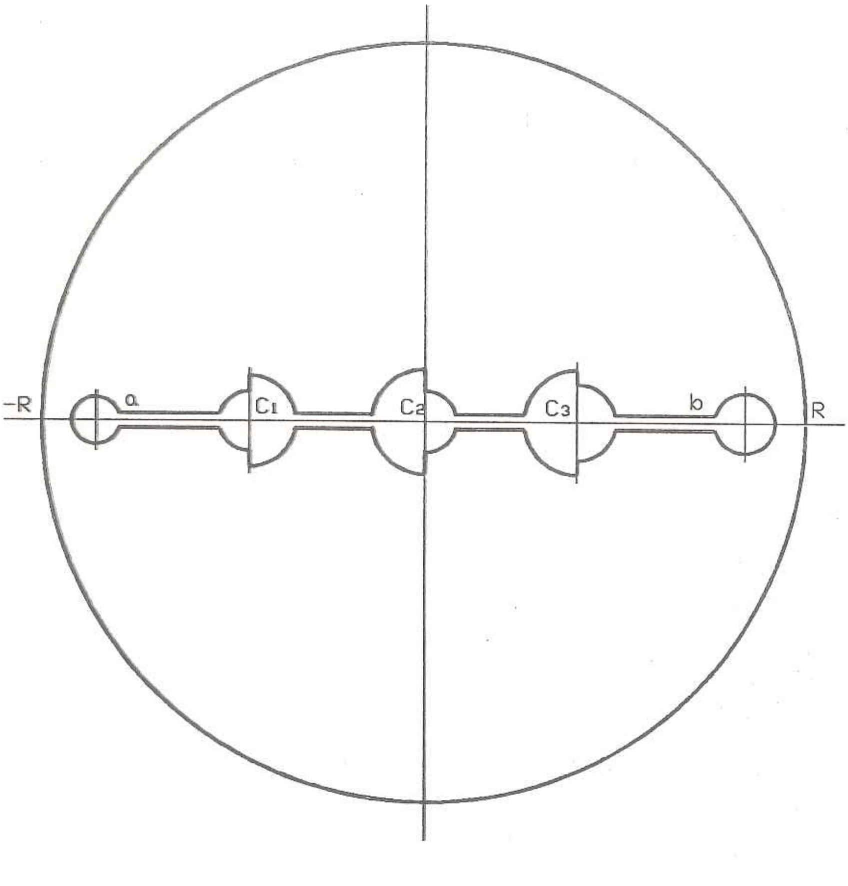
C1− ε11
C1+ ε12
C2 − ε21
C2 + ε22
R2


−R
a
C1
C2
C3
b
R

MAIN THEOREM. *Let*

1°. $b_{-1,\nu} = \mathrm{Res}_{C_\nu} f(z)$,

2°. $B_{-1,\nu} = \mathrm{Res}_{C_\nu}\left(f(z)\ln\frac{b-z}{z-a}\right)$,

3°. *Let* $r_{1\nu}, r_{2\nu}$ *satisfy conditions* (L, P, Q),

4°. *The complex logarithm* $\ln\frac{b-z}{z-a}$ *is defined in a usual way,* $\ln 1 = 0 + 0\pi i$.

Then the generalized value of the real integral depends on the following iterated boundary values

$$v.g.\int_a^b f(x)\,dx = {}_{(L,P,Q)^*}\int_a^b f(x)\,dx = \sum_{C\setminus[a,b]} \mathrm{Res}\, f(z)\ln\frac{b-z}{z-a} - \mathrm{Res}_{z=\infty} f(z)\ln\frac{b-z}{z-a}$$

$$+\lim_{r_{i1}\to 0}(\ldots\lim_{r_{i2}\to 0}\ldots\lim_{r_{jk}\to 0}\ldots\left\{\left(\sum_{\nu=1}^{n} b_{-1,\nu}\right)\left(-\pi i+\ln\frac{r_{1\nu}}{r_{2\nu}}\right)+\right.$$

$$+\frac{1}{2\pi i}\sum_{\nu=1}^{n}\sum_{k=2}^{n_\nu}\frac{1}{1-k}\left(B_{-k,\nu}+2\pi i b_{-k,\nu}\right)\left[\frac{e^{i(k-1)\pi/2}-1}{r_{2\nu}^{k-1}}+\right.$$

$$+\frac{e^{i\pi(k-1)}-e^{i(k-1)\pi/2}}{r_{1\nu}^{k-1}}+i^{1-k}\left(\frac{1}{r_{1\nu}^{k-1}}-\frac{1}{r_{2\nu}^{k-1}}\right)+$$

$$\left.\left.+\frac{B_{-k,\nu}}{k-1}\left(\frac{(-1)^{k-1}}{r_{1\nu}^{k-1}}-\frac{1}{r_{2\nu}^{k-1}}\right)\right]\right\}.$$

This formula contains explicitly all known results of the traditional calculus residius.

Example 1.

1°. If $f(x)$ has poles of order one then the principal value is

$$v.p.\int_a^b f(x)\,dx = \sum_{C\setminus[a,b]} \mathrm{Res}\, f(z)\ln\frac{b-z}{z-a} - \mathrm{Res}_{z=\infty} f(z)\ln\frac{b-z}{z-a} -$$

$$\pi i\sum_{\nu=1}^{n}\mathrm{Res}_{z=C_\nu}$$

(formula Dimitrovski-Adamović [2]).

2°. If the function $f(x)$ is continuous on $[a, b]$ then ordinary real integral is

$$\int_a^b f(x)\,dx = \sum_{C\setminus[a,b]} \mathrm{Res}\, f(z)\ln\frac{b-z}{z-a} - \mathrm{Res}_{z=\infty} f(z)\ln\frac{b-z}{z-a}$$

If $f(z)$ has property that $z^2 f(z)\to M$, $z\to\infty$, then

$$\int_a^b f(x)\,dx = \sum_{C\setminus[a,b]} \mathrm{Res}\, f(z)\ln\frac{b-z}{z-a} \qquad \text{(the Cauchy formula).}$$

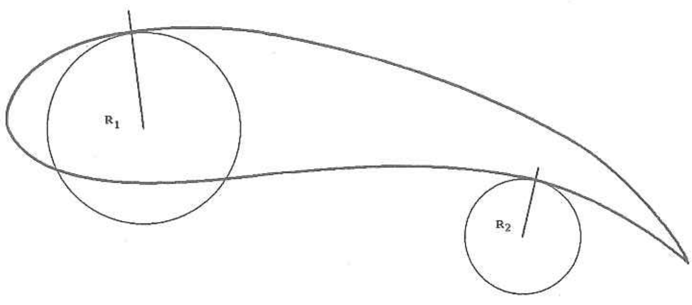
3°. Existence of principal value in the case of poles of higher order

THEOREM. *Necessary and sufficient condition for the existence of the principal value of the integral* $\int_a^b f(x)\,dx$ *is : the sum of the Laurent coefficients of the members* $(z-C_\nu)^{-2k}$ *with even orders, in the Laurent series in* C_ν, *is equal to zero, i.e.,*

$$b_{-2,1}+b_{-2,2}+\cdots+b_{-2,n}=0;\quad b_{-2m,1}+b_{-2m,2}+\cdots+b_{-2m,n}=0.$$

The principal value is given by the above formula.

Example 2. For the integral which we use to construct the plane wing we have:



Where $K_1=\frac{1}{R_1}$, $K_2=\frac{1}{R_2}$ are curvatures of the wing surfaces, (values ϵ_1,ϵ_2 dependent on K_1 and K_2.

$$v.g.^*\int_0^{\pi/2}\frac{d\phi}{\underset{(a<1)}{a-\cos\phi}}=\frac{1}{\sqrt{1-a^2}}\lim_{\epsilon_1\to0}\lim_{\epsilon_2\to0}\left\{\ln\left|\frac{\epsilon_1}{\epsilon_2}\right|+\ln\frac{\sqrt{1+a}-\sqrt{1-a}}{\sqrt{1+a}+\sqrt{1-a}}\right\},$$

The values $\epsilon_i\to0$ are with different speeds but the case $\epsilon_2=k\epsilon_1$ is possible. Then

$$\underset{(\epsilon_2=k\epsilon_1)}{v.g.}{}^*\int_0^{\pi/2}\frac{d\phi}{a-\cos\phi}=\ln\left|k\frac{\sqrt{1+a}-\sqrt{1-a}}{\sqrt{1+a}+\sqrt{1-a}}\right|^{1/\sqrt{1-a^2}}.$$

This formula is consequence of the our theorem (it is also elementary).

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