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INTERIOR HOMOGENEOUS BOUNDARY VALUE PROBLEM FOR SIMPLE CONECTED-REGIONS

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Abstract. *In this paper the functions $\Theta_1(z)$ and $\Theta_2(z)$ which are analytic in the simple connected regions S^+ and D^+, respectively, were determined when the boundary conditions*

$$\Theta_2(\beta(t)) = G(t) \cdot \Theta_1(t)$$

are known.

Let S^+ and D^+ be finite simple connected regions, bounded by closed Ljapunov curvs L and Γ respectively. Suppose that the boundaries L and Γ are traversed in the positive sense relative to their interiors S^+ and D^+ respectively, so that a person moving along L or Γ in this direction always has their interiors lying to his left.

Let $\beta(t)$ be the function given on L satisfying the following conditions:

a) It transforms homeomrphically the closed contour L into the closed contour changing the direction of movement.

b) Function $\beta(t)$ has continuous derivatives which are different from zero at all the points of the contour L.

Let the function $\beta^{-1}(t)$, $t \in L$, be the inverse function of $\beta(t)$.

We shall determine the functions $\Theta_1(z)$ and $\Theta_2(z)$ which are analytic in S^+ and D^+, respectively, whose boundary values on the appropriate contours satisfy the following boundary condition

$$\Theta_2(\beta(t)) = G(t) \cdot \Theta_1(t)\,,\ t \in L\,, \tag{1}$$

where $G(t)$ is a continuous function on L in the sense of Holder. First, suppose that $k = \dfrac{1}{2\pi}\left[\arg G(t)\right]_L = 0$, and consider the boundary value problem

$$\Gamma_2(\beta(t)) - \Gamma_1(t) = \ln G(t)\,,\ t \in L\,, \tag{2}$$

where the function $\ln G(t)$ satisfies the Holder's condition on L. It is known that the particular solution of (2) is determined by the formulas

$$\Gamma_1(z) = -\frac{1}{2\pi i}\int_L \frac{\sigma(t)}{t-z}\,dt\,, \qquad z \in S^+\,,$$

$$\Gamma_2(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\sigma(\beta^{-1}(t))}{t-z}\,dt\,, \qquad z \in D^+\,,$$

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where $\sigma(t)$, $(t \in L)$ is the solution of the Fredholm integral equation

$$(F\sigma)(t) \equiv \sigma(t) - \frac{1}{2\pi i}\int_L \left[\frac{1}{t-\tau} - \frac{\sigma'(\tau)}{\sigma(\tau)-\sigma(t)}\right]\sigma(\tau)\,d\tau = \ln G(t)\,,\; t,\tau \in L.$$

Now, it is easy to check that the coefficient $G(t)$ from the problem (2) we can represent in the form

$$G(t) = \frac{X_{0,1}(\beta(t))}{X_0(t)}$$

where $X_0(z) = e^{\Gamma_1(z)}$, $z \in S^+$ and $X_{0,1} = e^{\Gamma_2(z)}$, $z \in D^+$. In this way, the boundary condition (1) can be represented in the following way:

$$\frac{\Theta_2(\beta(t))}{X_{0,1}(\beta(t))} = \frac{\Theta_1(t)}{X_0(t)}, \quad t \in L.$$

So, the functions $\delta_2(z) = \dfrac{\Theta_2(z)}{X_{0,1}(z)}$, $z \in D^+$, and $\delta_1(z) = \dfrac{\Theta_1(z)}{X_0(z)}$, $z \in S^+$, satisfy the following boundary condition

(3) $$\delta_2(\beta(t)) = \delta_1(t)\,, \quad t \in L\,,$$

The general solution to the problem (3) is given by formulae $\delta_1(z) = C$ and $\delta_2(z) = C$ where C is an arbitrary complex constant. So, the functions $\Theta_2(z) = C \cdot e^{\Gamma_2(z)}$, $z \in D^+$ and $\Theta_1(z) = C \cdot e^{\Gamma_1(z)}$, $z \in S^+$, are the general solution to the problem (1) in the case $k = 0$.

Let us consider the boundary condition (1) where the index k corresponded to the function $G(t)$ is any real number. Assume that the coordinate origin belongs to the region S^+ and define the function $G_0(t)$ in the following way: $G_0(t) = t^{-k}G(t)$, $t \in L$.

Now, $\frac{1}{2\pi}[\arg G_0(t)]_L = 0$. Hence, for the homogeneous boundary value problem, with the coefficient $G_0(t)$ there exist the functions $X_0(z)$ and $X_{0,1}(z)$ being analytic in S^+ and D^+ respectively, and different from zero successively in $S^+ \cup L$ and $D^+ \cup \Gamma$, and which on the appropriate contours L and Γ have the limits $X_0(t) \in H(L)$ and $X_{0,1}(t) \in H(L)$ satisfying the following boundary value condition:

$$X_{0,1}(\beta(t)) = G_0(t) \cdot X_0(t)\,, \; t \in L.$$

Those functions are determined by the formulae

$$X_0(z) = \exp\Big[-\frac{1}{2\pi i}\int_L \frac{\sigma(t)}{t-z}\,dt\Big]\,, \qquad z \in S^+\,,$$

$$X_{0,1}(z) = \exp\Big[\frac{1}{2\pi i}\int_\Gamma \frac{\sigma(\beta^{-1}(t))}{t-z}\,dt\Big]\,, \qquad z \in D^+\,,$$

where $\sigma(t)$, $t \in L$, is the solution of the equation $(F\sigma)(t) = \ln G_0(t)$. According to all of this it follows that on the contour L, the coefficient $G(t)$ of the problem (1) can be represented in the form

(4) $$G(t) = \frac{X_{0,1}(\beta(t))}{t^{-k}X_0(t)}, \qquad t \in L\,,$$

From the relations (1) and (4) we get the following boundary conditions

$$\frac{\Theta_2(\beta(t))}{X_{0,1}(\beta(t))} = \frac{\Theta_1(t)}{t^{-k}X_0(t)}, \quad t \in L. \tag{5}$$

Let us denote by $f_2(z)$ the functions $\dfrac{\Theta_2(z)}{X_{0,1}(z)}$, $z \in D^+$, and distinct the cases $k < 0$ and $k \geq 0$.

a) **Let k < 0.**

The function $\dfrac{\Theta_1(z)}{z^{-k}X_0(z)}$, $z \in S^+$, has the point $z = 0$ as a pole of order $-k$, so that it can be represented in the form:

$$\frac{\Theta_1(z)}{z^{-k}X_0(z)} = \sum_{i=1}^{-k} \frac{c_i}{z^i} + f_1(z), \qquad z \in S^+,$$

where $f_1(z)$ is indefinite analytic function in S^+ and c_i, $i = 1, 2, \ldots, -k$, are complex constants.

If we introduce notation $B_{2j-1} = \operatorname{Re} c_j$, $B_{2j} = \operatorname{Im} c_j$,

$$\tau_{2j-1}(t) = \frac{1}{t^j}, \, t \in L, \qquad \tau_{2j}(t) = \frac{i}{t^j}, \, t \in L,$$

then by (3) we shall get

$$f_1(z) = B_0 - \sum_{j=1}^{-2k} \frac{B_j}{2\pi i} \int_L \frac{\sigma_j(t)}{t-z}\, dt, \qquad z \in S^+$$

$$f_2(z) = B_0 + \sum_{j=1}^{-2k} \frac{B_j}{2\pi i} \int_\Gamma \frac{\sigma_j(\beta^{-1}(t))}{t-z}\, dt, \qquad z \in D^+,$$

where B_0 is an arbitrary complex constant and $\sigma_j(t)$ are the solutions of Fredholm's integral equations

$$(F\sigma_j)(t) = \tau_j(t), \qquad j = 1, 2, \ldots, -2k.$$

Let us assume that $B_{2j+1} = \operatorname{Re} B_0$, $B_{2j+2} = \operatorname{Im} B_0$ and let us define the functions:

$$U_{2j-1}(z) = \frac{1}{z^j} - \frac{1}{2\pi i} \int_L \frac{\sigma_{2j-1}(t)}{t-z}\, dt, \qquad z \in S^+,$$

$$U_{2j}(z) = \frac{i}{z^j} - \frac{1}{2\pi i} \int_L \frac{\sigma_{2j}(t)}{t-z}\, dt, \qquad z \in S^+,$$

$$j = 1, 2, \ldots, -2k,$$

$$U_{-2k+1}(z) = 1, \quad z \in S^+, \qquad U_{-2k+2}(z) = i, \quad z \in S^+,$$

$$V_j(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\sigma_j(\beta^{-1}(t))}{t-z}\, dt, \quad z \in D^+, \quad j = 1, 2, \ldots, -2k,$$

$$V_{-2k+1}(z) = 1, \quad z \in D^+, \qquad V_{-2k+2}(z) = i, \quad z \in D^+.$$

Now, we can formulate the general solution of the homogeneous boundary value problem (1), in the case $k < 0$, in the following way

$$(6)\qquad \begin{aligned} \Theta_1(z) &= z^{-k} X_0(z) \sum_{i=1}^{-2k+2} B_i U_i(z)\,, && z \in S^+\,, \\ \Theta_2(z) &= X_{0,1}(z) \sum_{i=1}^{-2k+2} B_i V_i(z)\,, && z \in D^+\,. \end{aligned}$$

b) Let $k \geq 0$.

Now, the function $\dfrac{\Theta_1(z)}{z^{-k} X_0(z)}$ is analytic in S^+ and for $k > 0$ is equal to zero at the coordinate origin, and the function $\dfrac{\Theta_2(z)}{X_{0,1}(z)}$ is analytic in D^+. According to (5) we get that the solution of the problem (1) can be presented in the form

$$\begin{aligned} \Theta_1(z) &= C \cdot z^{-k} \cdot X_0(z)\,, && z \in S^+\,, \\ \Theta_2(z) &= C \cdot X_{0,1}(z)\,, && z \in D^+\,, \end{aligned}$$

where C is an arbitrary complex constant. For $z = 0$ it is easy to verify that $C = 0$ and consequently $\Theta_1(z) = 0$, $z \in S^+$, and $\Theta_2(z) = 0$, $z \in D^+$.

If $k = 0$ then $\Theta_1(z) = C \cdot X_0(z)$, $z \in S^+$, and $\Theta_2(z) = C \cdot X_{0,1}(z)$, $z \in D^+$. This solution can be obtained from the formula (6) assuming that (6) holds for $k = 0$. Therefore, we have proved the following theorem:

THEOREM *If the index $k \leq 0$, then the boundary value problem (1) is solvable and its solution can be represented by the formula (6) containing $2(-k+1)$ arbitrary real constants. If, however, $k > 0$ then problem (1) has only trivial solution.*

REFERENCES

[1] GAHOV F. D., *Krajevije zadači*, Nauka, 1977.
[2] ISOHANOV R. S., *O nekatorih graničnih zadač teoriji analitičeskih funkcij*, Trudi Mat. in-ta AN Gruz. SSSR, 52 (1976), 61-80.
[3] NEČAJEV A. H., *Pro adnu krajevuju zadaču dlja pari funkcij analitičeskih v oblasti*, Donov. AN URSR, 10 (1979), 891-893.

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