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A FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS IN PARANORMED SPACES

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ABSTRACT *The purpose of this paper is to generalized the fixed point theorem for multivalued mappings proved in* [2] *for a class of subsets of paranormed spaces.*

1. Introduction. Let E be a linear space on the real or complex number field. The function $|| \, ||^* : E \mapsto [0, +\infty)$ will be called paranorm iff:

1. $||x||^* = 0 \Leftrightarrow x = 0$;
2. $||-x||^* = ||x||^*$, for every $x \in E$;
3. $||x + y||^* \leq ||x||^* + ||y||^*$, for every $x, y \in E$;
4. If $||x_n - x_0||^* \to 0, \lambda_n \to \lambda_0$ *then* $||\lambda_n x_n - \lambda_0 x_0||^* \to 0$, $n \to \infty$.

The function $d : E \times E \to [0, +\infty)$ defined by $d(x, y) = ||x - y||^*$ is the distance function on E , and $(E, || \, ||^*)$ is a topological vector space.

DEFINITION 1. *The subset K of $(E, || \, ||^*)$ is said to be of Zima's type iff there exists a number $C = C(K) > 0$ such that*

$$||\lambda x||^* \leq C \cdot \lambda \cdot ||x||^*$$

for every $0 \leq \lambda \leq 1$ and every $x \in K - K$

REMARK. O. Hadžić [1] gave an example of $K(K \subseteq E)$, where $(E, || \, ||^*)$ is not a locally convex paranormed space, such that K is of Zima's type.

DEFINITION 2. *A subset K of a metric space (X, d) is called proximal iff for each $x \in X$, there exists an element $k \in K$ such that $d(x, k) = d(x, K)$ where $d(x, K) = \inf(d(x, y)| \;\; y \in K\}$.*

We denote the family of all nonempty bounded proximal subsets of X by 2^X_{bp} and the Hausdorff metric defined on 2^X_{bp} induced by d by H, i.e., for $A, B \in 2^X_{bp}$, $H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\}$.

Let $T : X \to 2^X_{bp}$. Then, for $x \in X$, by an orbit of x under T , $\sigma(x)$, we mean the sequence $\{x_n : x_0 = x, x_n \in Tx_{n-1}\}$. An orbit $\sigma(x)$ is called a strongly regular if

$$\sigma(x) = \{x_n | x_n \in Tx_{n-1}, d(x_n, x_{n-1}) = d(x_{n-1}, Tx_{n-1})\}$$

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DEFINITION 3. *A convex metric space (X, d) have Property (C) iff every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection.*

REMARK. Every weakly compact convex subset of a Banach space has Property (C).

DEFINITION 4. *The convex hull of a set $A(A \subset E)$ is the intersection of all convex sets in E containing A and it is denoted by conv A.*

2. Results

THEOREM 1 *Let $(E, \| \|^*)$ be a complete paranormed space and K a nonempty closed bounded convex subset of E with Property (C). Let T be a mapping of K into the family of nonempty convex proximal subsets of K such that $T(K)$ is of Zima's type and*

$$H(Tx, Ty) \leq \phi(\frac{1}{k} \max\{d(x, Tx), d(y, Ty)\}),$$

for $k = \max\{1, C(T(K))\}$, each $x, y \in E$ and $\phi : [0, +\infty) \to [0, +\infty)$ nondecrea sing right continuous function such that $\phi(t) < t$ for $t > 0$. Then there exists a nonempty subset M of K such that $T\,x = M$ for all $x \in M$.

PROOF. For any $x_0 \in K$ we may construct a strongly regular orbit at x_0 for T. First we claim that $\lim d(x_n, Tx_n) = 0$ where $\sigma(x_0) = \{x_n\}$. Observe that

$$D_n \equiv d(x_n, Tx_n) \leq H(Tx_{n-1}, Tx_n) \leq \phi\,(\frac{1}{k} \max\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\})$$

so that if $D_{n-1} < D_n$, then $D_n \leq \phi(\frac{1}{k} D_n) < D_n$. This is a contradiction. Thus $D_n \leq \phi(\frac{1}{k} D_{n-1}) < D_{n-1}$. Since $\{D_n\}_{n \in N}$ is a monotone decreasing sequence of non-negative real numbers, $\lim_{n\to\infty} D_n = D$ exists. If $D > 0$, then using the right continuity of ϕ we obtain

$$D \leq \lim_{D_n \to D+0} \phi(D_n) = \phi(D) < D$$

This contradiction shows that $D = 0$.

Now we let $H_\varepsilon = \{x \mid d(x, Tx) \leq \varepsilon\}$ for each $\varepsilon > 0$. From the above argument we have that $H_\varepsilon \neq \phi$ for each $\varepsilon > 0$.

Our second claim is that $\overline{\text{conv}} T(H_\varepsilon) \subseteq H_\varepsilon$ for each $\varepsilon > 0$. Let $y \in \overline{\text{conv}}\, T(H_\varepsilon)$ and let $\delta > 0$ be given: Choose $\lambda_i \in [0, 1]$ $y_i \in H_\varepsilon$ *and* $y_i^* \in Ty_i$ for each $i = 1, 2, \dots, n$ so that $\sum_{i=1}^n \lambda_i = 1$ and

$$d(y, \sum_{i=1}^{n} \lambda_i y_i^*) \leq \delta.$$

Since Ty is proximal, there exists $z_i \in Ty$ such that $d(y_i^*, z_i) = d(y_i^*, Ty)$ for $i = 1, 2, \dots, n$.

Now

$$\begin{aligned}
d(y,Ty) &\le d(y,\sum_{i=1}^{n}\lambda_i y_i^*) + d(\sum_{i=1}^{n}\lambda_i y_i^*, Ty)l) \\
&\le \delta + d(\sum_{i=1}^{n}\lambda_i y_i^*, \sum_{i=1}^{n}\lambda_i z_i) \le \delta + C(T(K))\sum_{i=1}^{n}\lambda_i d(y_i^*, z_i) \\
&= \delta + C(T(K))\sum_{i=1}^{n}\lambda_i d(y_i^*, Ty) \le \delta + C(T(K))\sum_{i=1}^{n}\lambda_i H(Ty_i, Ty) \\
&\le \delta + C(T(K))\sum_{i=1}^{n}\lambda_i \phi(\frac{1}{k}\max\{d(y_i,Ty_i), d(y,Ty)\}) \\
&\le \delta + k\phi(\frac{1}{k}\max\{\varepsilon, d(y,Ty)\}).
\end{aligned}$$

If $d(y,Ty) > \varepsilon$, than $d(y,Ty) \le \delta + k\phi(\frac{1}{k}d(y,Ty))$. Since $\delta > 0$ is arbitrary, this leads to an obvious contradiction that $d(y,Ty) \le k\phi(\frac{1}{k}d(y,Ty)) < d(y,Ty)$. Hence we must have $d(y,Ty) \le \varepsilon$ and $y \in H_\varepsilon$. This prove our second claim.

Let $\mu = \{\overline{\text{conv}}\ T(H_\varepsilon) \mid \varepsilon > 0\}$. Then μ is a bounded decreasing net of nonempty closed convex subsets so by Property (C) it has nonempty intersection. Hence $\phi \ne \cap\mu \subseteq \cap\{H_\varepsilon \mid \varepsilon > 0\}$. This shows that function $x \to d(x,Tx)$ attains it's infinum over K and because of the first claim this infinum must be zero. Let $M = \cap\{H_\varepsilon \mid \varepsilon > 0\}$ and proof is complete.

Using the proof of Theorem 1 and Theorem 2 [2] one can prove.

THEOREM 2. *Let $(E, \|\ \|^*)$ be a complete paranormed space and K a nonempty closed bounded convex subset of E with Property (C). Let T be a mapping of K into the family of nonempty convex proximal subsets of K such that $T(K)$ is of Zima's type and T satisfies condition:*
For given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in K$

$$\varepsilon \le \max\{d(x,Tx), d(y,Ty)\} < \varepsilon + \delta \Rightarrow H(Tx,Ty) < \frac{\varepsilon}{k}$$

Then there exists a nonempty subsets M of K such that $Tx = M$ for all $x \in M$.

REFERENCES

[1] O. HADŽIĆ, *On equilibrium point in topological vector spaces*, Comm. Math. Univ. Carolinea, 23 (1982), 727-738.

[2] H. KANEKO, *A report on general contractive type conditions for multi-valued mappings*, Math. Japonica, 33(4) (1988), 142-149.

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