Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 6 (1992), 49–54

FILOMAT–20, Niš, September 26–28, 1991

FUZZY RANDOM VARIABLE

MILA STOJAKOVIĆ

ABSTRACT. *This paper deals with fuzzy-set valued mappings of an R^n space whose values are closed, normal and compactly supported fuzzy sets. We study integrability and conditional expectation of such functions and give an existence and uniqueness theorem for fuzzy martingales.*

1. Introduction. The concept of a fuzzy set was introduced by Zadeh (1965). Puri, Ralescu, Klement and the others studied fuzzy random variable as a generalization of random sets. Fuzzy random variables are random variables whose values are not real numbers, as usually is the case, but fuzzy sets. A fuzzy set may assume different values of R^n, with each of which a degree of acceptability is associated. These degrees of acceptability are considered as truth values, and are handled according to the rules of fuzzy logic.

In Section 3 we review certain properties of fuzzy variables and their relationship to fuzzy sets and random sets. In Section 4 we define the fuzzy conditional expectation and investigate its properties. In Section 5 we prove a theorem for fuzzy martingales.

2. Preliminaries. In this paper we restrict our attention to the set of fuzzy random variables on the base space R^n, adapting in what follows definitions and results from Feron [5] and Puri, Ralescu [12]. A fuzzy set $u \in \mathcal{F}(R^n)$ is a function $u : R^n \to [0,1]$ for which

1. $u_0 = \overline{co}\{x \in R^n ; u(x) > 0\}$ is compact,
2. the α-level set u_α of u, defined by

$$u_\alpha = \{x \in R^n : u(x) \geq \alpha\}$$

is nonempty, closed and convex subset of R^n for all $\alpha \in (0,1]$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space where P is a probability measure. A fuzzy random variable is a function $X : \Omega \to \mathcal{F}(R^n)$ such that

$$\{(\omega, x) : x \in (X(\omega))_\alpha\} \in \mathcal{A} \times \mathcal{B}, \text{ for every } \quad \alpha \in [0,1],$$

Supported by the Science Fund of Serbia, grant number 0401A, through Matematički institut.

where $\mathcal{B}$ denotes the Borel subsets of R^n.

It is obvious that the function $X_\alpha : \Omega \to 2^{R^n}$ defined by $X_\alpha(\omega) = (X(\omega))_\alpha$ is the R^n-valued random set. If H is Hausdorff metric defined on $\mathcal{P}(R^n)$ (the space of all compact and convex subsets of R^n)

$$H(A, B) = \max\{\sup_{x\in A}\inf_{y\in B}||x-y||,\ \sup_{y\in B}\inf_{x\in A}||x-y||\},\ \ A, B \in \mathcal{P}(R^n),$$

then $(\mathcal{P}(R^n), H)$ is a complete metric space.

For any multifunction $F : \Omega \to \mathcal{P}(R^n)$ we can define the set

$$S_F = \{f \in L(\Omega, \mathcal{A}) : f(\omega) \in F(\omega) \quad P-a.e.\}$$

where $L(\Omega, \mathcal{A}) = L$ denotes the set of all functions $h : \Omega \to R^n$ which are integrable with respect to the probability measure P.

The set $S_F \subset L$ is closed with respect to a norm in L defined by

$$||h|| = \int_\Omega ||h(\omega)||dP, \quad h \in L.$$

Using S_F we can now define an integral for F (first introduced by Aumann [1])

$$\int_\Omega FdP = \{\int_\Omega f(\omega)dP(\omega) : f \in S_F\}.$$

The integrals $\int_\Omega f(\omega)dP(\omega)$ are defined in the sense of Bochner. $F : \Omega \to \mathcal{P}(R^n)$ is called integrably bounded if there exists integrable real valued function $h : \Omega \to R$ such that $\sup_{x\in F(\omega)} ||x|| \le h(\omega)$ $P-a.e.$ The fuzzy random variable $X : \Omega \to \mathcal{F}(R^n)$ is integrably bounded if X_α is integrably bounded for all $\alpha \in [0,1]$. Let $\mathcal{L} = \mathcal{L}(\Omega, \mathcal{A})$ denotes the set of all integrably bounded multivalued functions $F : \Omega \to \mathcal{P}(R^n)$ and let $\Lambda = \Lambda(\Omega, \mathcal{A})$ be the set of all integrably bounded fuzzy random variables $X : \Omega \to \mathcal{F}(R^n)$.

We shall close this section, by recalling a lemma which we shall use in the sequel.

LEMMA 1 ([10]). *Let M be a set and let $\{M_\alpha : \alpha \in [0,1]\}$ be a family of subsets of M such that*

1. $M_0 = M$
2. $\alpha \le \beta \Rightarrow M_\beta \subseteq M_\alpha$
3. $\alpha_1 \le \alpha_2 \le \dots, \lim_{n\to\infty} \alpha_n = \alpha \Rightarrow M_\alpha = \bigcap_{n=1}^{\infty} M_{\alpha_n}$.

Then, the function $\phi : M \to [0,1]$ defined by $\phi(x) = sup\{\alpha \in [0,1] : x \in M_\alpha\}$ has the property that $\{x \in M : \phi(x) \ge \alpha\} = M_\alpha$ for every $\alpha \in [0,1]$.

For all $X, Y \in \Lambda$ we can define the function $\mathcal{D} : \Lambda \times \Lambda \to R$

$$\mathcal{D}(X, Y) = \sup_{\alpha\ge 0} \Delta(X_\alpha, Y_\alpha).$$

Two fuzzy variables $X, Y \in \Lambda$ are considered to be identical if $\mathcal{D}(X, Y) = 0$. It is obvious that $\mathcal{D}$ is a metric in Λ since Δ is metric in $\mathcal{L}$ (Th. 3.3 [6]).

THEOREM 1 ([14]). *$(\Lambda, \mathcal{D})$ is a complete metric space.*

3. Expectation of fuzzy random variable. If X is a fuzzy random variable from Λ, we can define the family of subsets of R^n by

$$M_\alpha = \int_\Omega X_\alpha dP, \quad \alpha \in (0,1].$$

We know that X_α is closed valued integrably bounded random set which implies that $M_\alpha \neq \emptyset$ and M_α is compact for all $\alpha \in (0,1]$. We shall show that the family $\{M_\alpha\}_{\alpha\in(0,1]}$ define a fuzzy set from $\mathcal{F}(R^n)$. 1 and 2 from Lemma 1 are satisfied. To prove that for nondecreasing sequence $\{\alpha_i\}$, $\lim_{i\to\infty} \alpha_i = \alpha > 0$ implies $M_\alpha = \cup_{i=1}^\infty M_{\alpha_i}$ we proceed as follows: The sequence $X_{\alpha_1}, X_{\alpha_2}, \dots$ is measurable, integrably bounded by $h_{\alpha_1} \subset L$, i.e.

$$\sup_{x\in X_{\alpha_i}(\omega)} \|x\| \leq \sup_{x\in X_{\alpha_1}(\omega)} \|x\| \leq h_{\alpha_1}(\omega) \text{ for all } \omega \in \Omega.$$

From $\alpha \geq \alpha_i$ we get

$$M_\alpha = \int_\Omega X_\alpha dP \subseteq \int_\Omega X_{\alpha_i} dP = M_{\alpha_i}$$

for all $i \in N$. From the compactness of M_β, $\beta \in (0,1]$, it follows

$$M_\alpha = \int_\Omega X_\alpha dP \subseteq \cap_{i=1}^\infty \int_\Omega X_{\alpha_i} dP = \lim_{i\to\infty} \int X_{\alpha_i} dP.$$

The compactness of $X_{\alpha_i}(\omega)$ implies that

$$X_{\alpha_i}(\omega) \to X_\alpha(\omega), \text{ for all } \omega \in \Omega,$$

that is,

$$\lim H(X_{\alpha_i}(\omega), X_\alpha(\omega)) = 0 \text{ for all } \omega \in \Omega.$$

From properties of Auman's integral, we get

$$\begin{aligned}\lim_{i\to\infty} H\Big(\int_\Omega X_\alpha dP, \int_\Omega X_{\alpha_i} dP\Big) &\leq \lim_{i\to\infty} \Delta(X_\alpha, X_{\alpha_i}) \\ &= \lim_{i\to\infty} \int_\Omega H(X_\alpha(\omega), X_{\alpha_i}(\omega)) dP\end{aligned}$$

and using classical Lebesgue dominated theorem, we obtain

$$\int_\Omega X_\alpha dP \xrightarrow{H} \int_\Omega X_{\alpha_i} dP.$$

So, we have shown that the family $\{M_\alpha\}_{\alpha\in(0,1]}$ satisfies all the conditions of Lemma 1 which means that it defines one and only one fuzzy set. For all $\alpha \in (0,1]$ $X_\alpha \subseteq X_0$ and

$$M_\alpha = \int_\Omega X_\alpha dP \subseteq \int X_0 dP$$

which implies that

$$\cup_{\alpha\in(0,1]} M_\alpha \subseteq \int X_0 dP \Rightarrow M_0 = \overline{co}\cup_{\alpha\in(0,1]} M_\alpha \text{ is compact }.$$

The fuzzy set defined by family $\{M_\alpha\}$ we shall call integral or expectation of fuzzy random variable $X \subset \Lambda$ and denote by

$$\int X dP.$$

Hence we can formulate the next theorem.

Theorem 2 *If $X : \Omega \to \mathcal{F}(R^n)$ is an integrably bounded fuzzy random variable, then there exists a unique fuzzy set $u \in \mathcal{F}(R^n)$ such that*

$$u_\alpha = \int X_\alpha dP \textit{ for all } \alpha \in (0,1].$$

4. Fuzzy conditional expectation. Motivated by the definition of conditional expectation for a random set we introduce the notion of fuzzy conditional expectation for fuzzy random variable.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{F}$ a sub–σ–algebra of $\mathcal{A}$ and $F \in \mathcal{L}$. The conditional expectation of F with respect to $\mathcal{F}$, which is in $\mathcal{L}(\Omega,\mathcal{F})$, is determined by setting

$$S_{E(F|\mathcal{F})} = cl\{g \in L(\Omega,\mathcal{F}) \,:\, g = E(f|\mathcal{F}), \quad f \in S_F\}.$$

Finally if X is a fuzzy random set we can define the conditional expectation of $X \in \Lambda$ in such a way that the following conditions are satisfied:

$$E(X|\mathcal{F}) \in \Lambda(\Omega,\mathcal{F}),$$

$$\{x \in R^n \,:\, E(X|\mathcal{F})(\omega)(x) \geq \alpha\} = E(X_\alpha(\omega)\,|\mathcal{F}).$$

The next theorem shows that there exists a unique fuzzy random variable satisfying these requirements. The proof is based on Lemma 1

Theorem 3 ([14]). *If $X \in \Lambda(\Omega,\mathcal{A})$, then there exists a unique fuzzy random variable $Y \in \Lambda(\Omega,\mathcal{F})$ such that*

$$Y_\alpha(\omega) = E(X_\alpha(\omega)\;|\mathcal{F}).$$

THEOREM 4 *If $X \in \Lambda$, then:*

$$\int_A E(X \mid \mathcal{F})dP = \int_A X dp, \;\; A \in \mathcal{F}.$$

PROOF. Since X_α is integrably bounded random set we have that

$$\int_A (E(X \mid \mathcal{F}))_\alpha dP = \int_A X_\alpha dP, \;\; A \in \mathcal{F}.$$

Equality $\int_A Y_\alpha dP = (\int_A Y dP)_\alpha$ for all $Y \in \Lambda$, implies

$$(\int_A E(X \mid \mathcal{F})dP)_\alpha = \int_A (E(X \mid \mathcal{F}))_\alpha dP = \int_A X_\alpha dP = (\int_A X dP)_\alpha$$

for all $\alpha \in (0,1]$, which means that

$$\int_A E(X \mid \mathcal{F})dP = \int X dP, \;\; A \in \mathcal{F}.$$

THEOREM 5. *The fuzzy conditional expectation has the following properties:*

1. $\mathcal{D}(E(X_1|\mathcal{F}), E(X_2|\mathcal{F})) \leq \mathcal{D}(X_1, X_2)$ *for all* $X_1, X_2 \in \Lambda$.

2. *If* $\mathcal{F}_1 \subset \mathcal{F} \subset \mathcal{A}$ *and* $X \in \Lambda$, *then* $E(X|\mathcal{F}_1)$ *taken on the base space* $(\Omega, \mathcal{A}, P)$ *is equal to* $E(X|\mathcal{F}_1)$ *taken on the base space* $(\Omega, \mathcal{F}, P)$.

3. *If* $\mathcal{F}_1 \subset \mathcal{F} \subset \mathcal{A}$ *and* $X \in \Lambda$, *then* $E(E(X|\mathcal{F})|\mathcal{F}_1) = E(X|\mathcal{F}_1)$.

4. *If* $X_n : \Omega \to \mathcal{F}(R^n)$ *are uniformly integrable bounded and* $X_n \xrightarrow{\mathcal{D}} X$, *then* $E(X_n|\mathcal{F}) \xrightarrow{\mathcal{D}} E(X|\mathcal{F})$.

This theorem is a fuzzy generalization of Th. 5.3. [6]. The proof is quite similar to the case of random sets so it is omitted.

Let $\{\mathcal{F}_n\}_{n \in N}$ be an increasing sequence of sub-σ-fields of $\mathcal{A}$ and let $\{X_n\}_{n \in N}$ be a sequence of integrably bounded fuzzy random variables adapted to $\{\mathcal{F}_n\}_{n \in N}$. Then, in analogy to the single valued and multivalued cases, we can introduce the following notations.

(1) The system $\{X^n, \mathcal{F}^n\}_{n \in N}$ is said to be a fuzzy valued martingale if and only if for all $n \geq 1$

$$E(X^{n+1} \mid \mathcal{F}^n)(\omega) = X^n(\omega) \text{ P - a.e.}$$

(2) The system $\{X^n, \mathcal{F}^n\}_{n \in N}$ is said to be a fuzzy submartingale (resp. supermartingale) if and only if

$$E(X^{n+1} \mid \mathcal{F}^n)(\omega) \supseteq X^n(\omega) \quad (respE(X^{n+1} \mid \mathcal{F}^n)(\omega) \subseteq X^n(\omega))P - a.e.$$

By $\mathcal{F}^\infty$ we shall denote the σ-algebra generated by $\cup_{n=1}^{\infty}\mathcal{F}^n$. In applications it is usually assumed that $\mathcal{F}^\infty = \mathcal{A}$. If $X \subset \Lambda$ then $\{E(X \mid \mathcal{F}^n), \mathcal{F}^n\}_{n\in N}$ forms a fuzzy valued martingale by Theorem 2.

THEOREM 6 *Let $X \subset \Lambda$ and let the fuzzy martingale $\{E(X \mid \mathcal{F}^n), \mathcal{F}^n\}$ be such that $\Delta(E(X_\alpha \mid \mathcal{F}^n), X_\alpha^\infty) \to 0$ uniformly for $\alpha \in (0,1]$ where $X_\alpha^\infty = E(X_\alpha \mid \mathcal{F}^\infty)$. Then*

$$D(X^n, X^\infty) \to 0,$$

where $X^\infty = E(X \mid \mathcal{F}^\infty)$.

REFERENCES

[1] R. J. AUMANN, *Integrals of set-valued functions,* J. Math. Anal. Appl., 12 (1965), 1-12.

[2] BOSWEL S., TAYLOR M., *A central limit theorem for fuzzy random variables,* Fuzzy Sets and Systems, 24 (1987), 331-334.

[3] CHATTERYI S.D., *A note on the convergence of Banach space valued martingales,* Math. Ann., 153 (1964), 142-149.

[4] DIAMOND P, KLOEDEN P., *Characterization of compact subsets of fuzzy sets,* Fuzzy Sets and Systems , 29 (1989), 341-348.

[5] R. FÉRON, *Ensembles aléatoires flous.,* C.R. Acad. Sci. Paris Ser., A 282 (1976), 903-906.

[6] F. HIAI, H. UMEGAKI, *Integrals, conditional expectation and martingales of multivalued functions,* J. Multivariate Anal., 7(1977), 149-182.

[7] R. KRUSE, R. BUCK-EMDEN, R. CORDES, *Processor power consideration - an application of fuzzy Markov chains,* Fuzzy Sets and Systems, 21 (1987), 289-299.

[8] R. KRUSE, *The strong law of large numbers for fuzzy random variables,* Information Sciences, 28 (1982), 233-241.

[9] H. KWAKERNAAK, *Fuzzy random variables. Definition and theorems,* Inform. Sci. 15 (1978), 1-29.

[10] C.V. NEGOITA, D. RALESCU, *Applications of fuzzy sets to systems analysis,* Wiley, New York, 1975.

[11] N. PAPAGEORGIOU, *On the theory of Banach space valued multifunctions,* J. Multivariate Anal. 17 (1985), 185-227.

[12] M.L. PURI, D.A. RALESCU, *Fuzzy random variables,* J.Math. Anal. Appl., 114 (1986), 406-422.

[13] M. STOJAKOVIĆ, LJ. GAJIĆ, *Fuzzy Markov chains,* Rew. of Research, FTN, (1990), 126-131.

[14] M. STOJAKOVIĆ, *R^n-valued fuzzy random variable,* Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 20, 2(1990), 95-105.

[15] UHL. J.J. JR., *The range of a vector valued measure,* Proc. Amer. Math. Soc., 23 (1969), 158-163.

[16] UMEGAKI H. AND BHARUCHA-RAID A.T., *Banach space valued random variables and tensor products of Banach spaces,* J. Math. Anal. Appl., 31 (1970), 49-67.

Faculty of Engineering, Department of Mathematics,
Trg D.Obradovića 7,
21 000 Novi Sad, Yugoslavia