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ON A THEOREM OF GONCHAR AND RAHMANOV

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Abstract. *In this note we give the complete proof of the theorem of Gonchar and Rahmanov* [2, *Теорема*].

Let

$$f(z)=\sum_{n=0}^{\infty} c_n z^n \tag{1}$$

be a holomorphic function in $z=0$ or formal power series on z. Let P_n be the set of polynomial in z with deg $\le n$, and $R_{n,m}=\{\frac{p}{q} : p\in P_n,\ q\in P_m,\ q\neq 0\}$

For integers n and m there are $p_{n,m}\in P_n$ and $q_{n,m}\in P_m$, $q\neq 0$, such that

$$(q_{n,m}f-p_{n,m})(z)=O(z^{n+m+1}),\qquad z\to 0. \tag{2}$$

Set

$$\pi_{n,m}(z)=\frac{p_{n,m}(z)}{q_{n,m}(z)}. \tag{3}$$

$\pi_{n,m}$ are called the Pade's approximations of type $[n/m]$ for (1) and $\{\pi_{n,m}\}_{n,m}=0$ is Pade's table for (1). Let $\pi_n(z)=\pi_{n,n}(z),\quad n=0,1,2,\dots$ be the diagonal Pade's approximates [1].

Let $H(U)$ be the set of holomorphic functions on the unit ball $U=\{z\in C : |z|<1\}$. If $f\in H(U)$ then π_n does not converges to f on U [6] (obviously because of poles).

If

$$\pi_n\in H(U), \tag{4}$$

then by [2]

$$\pi_n(z)\to f(z),\quad \text{uniformly on}\quad U. \tag{5}$$

Now, suppose that (4) is not true for all $n \geq n_0$, but only for some subsequence $\Lambda = \{n_k\}_{k=1}^{\infty}$. Of course (5) is not true. Let ρ be the radius of the maximal ball $U_\rho = \{\, z\, :\, |z| < \rho\}$ such that from (4) for $n \in \Lambda$, follows (5) when $n \to \infty$, $n \in \Lambda$. It is known that $\rho \leq 4/5$ [6, Теорема].

The best known lower bound for ρ has been established by

THEOREM OF GONCHAR AND RAHMANOV ([2,Теорема]). *Let $g(z,t)$ be the Green's function on $\{e^{i\theta} : \frac{\pi}{3} \leq \theta \leq \pi\}$, and $r \in (0,1)$ be the solution of the equation*

$$g(r,1) = 2g(r,0).$$

Then $\rho \geq r = 0,629$.

In [2], there is only the sketch of proof of [2,Теорема]. In this note we give the complete proof. We start with

LEMMA 1. *Let λ_1, be the Borel measure, $s(\lambda_1)$ support of λ_1 and $\omega_1 = \ln \frac{4}{3\sqrt{3}}$ be the Rabben's constant [5]. Then $s(\lambda_1) = \Gamma_{\frac{\pi}{3}}$, where $\Gamma_{\frac{\pi}{3}} = \{z = e^{i\theta} : \frac{\pi}{3} \leq |\theta| \leq \pi\}$.*

PROOF. For $z = e^{i\varphi}$, we have that $\ln \dfrac{1}{|1-z|} = \ln \dfrac{1}{2\sin\frac{\varphi}{2}}$ is convex function of φ, $\varphi \in (0, 2\pi)$; Let ν be Borel measure and V^ν logarithmic potential of ν [4,5], then

$$V^\nu(z) = \int \ln \frac{1}{\sin\frac{\varphi-\theta}{2}}\, d\mu(e^{i\theta})$$

is convex function of φ, $\varphi \in \partial U \setminus s(\nu)$.

Set $\Gamma_\theta = \{z = e^{i\varphi} : \theta \leq |\varphi| \leq \pi\}$, $(\theta < \pi)$.We shall prove that $s(\lambda_1) = \Gamma_{\theta_0}$, $0 < \theta_0 < \pi$. If it is not true, then there are $\theta', theta'' \in \Gamma_{\theta_0}$, $(\theta', \theta'') \cap s(\lambda_1) = \emptyset$, $\theta' \in s(\lambda_1)$, $\theta'' \in s(\lambda_1)$. From [5, Теорема 4.4] we have

$$V^{\lambda_1}(e^{i\theta'}) + \ln \frac{1}{|1-e^{i\theta'}|} = \omega_1 = V^{\lambda_1}(e^{i\theta''}) + \ln \frac{1}{|1-e^{i\theta''}|}.$$

For $\theta \in (\theta', \theta'')$ we have

$$V^{\lambda_1}(e^{i\theta}) + \ln \frac{1}{|1-e^{i\theta}|} \geq \omega_1.$$

Since, $V^{\lambda_1}(z)$ and $\ln \frac{1}{|1-z|}$ are lower convex functions on (θ', θ''), it follows that

$$V^{\lambda_1}(e^{i\theta}) + \ln \frac{1}{|1-e^{i\theta}|} = \omega_1, \qquad \theta \in (\theta', \theta'').$$

Thus

$$\frac{d}{d\theta}\left(V^{\lambda_1}(e^{i\theta}) + \ln \frac{1}{|1-e^{i\theta}|}\right) = 0, \qquad \theta \in (\theta', \theta''),$$

and so
$$\frac{d^2}{d\theta^2}\left(V^{\lambda_1}(e^{i\theta}) + \ln\frac{1}{|1-e^{i\theta}|}\right) = 0, \qquad \dot{\theta} \in (\theta', \theta'').$$
This is a contradiction.

Hence $s(\lambda_1) = \Gamma_{\theta_1,\theta_2} = \{e^{i\theta} : \theta_1 < \theta \le 2\pi - \theta_2\}$. We shall prove that $\theta_1 = \theta_2$. If $\theta_1 \ne \theta_2$, set $\lambda_1^*(e) = \lambda_1(\bar{e})$, e is a Borel subset of ∂U, $\bar{e} = \{\bar{z} : z \in e\}$. Now

$$V^{\lambda_1}(z) + \ln\frac{1}{|1-z|} = \quad \begin{cases} = \omega_1 & z \in \Gamma_{\theta_1,\theta_2} \\ \ge \omega_1 & |z| = 1 \end{cases},$$

but

$$V^{\lambda_1}(z) + \ln\frac{1}{|1-z|} = V^{\lambda_1^*}(\bar{z}) + \ln\frac{1}{|1-\bar{z}|} = \quad \begin{cases} = \omega_1 & z \in \Gamma_{\bar{\theta}_1,\bar{\theta}_2} \\ \ge \omega_1 & |z| = 1 \end{cases}.$$

Since, $\lambda_1^* = \lambda_1$ ([5,Утверждение 4.3]) we have $\theta_1 = \theta_2$, and so $s(\lambda_1) = \Gamma_{\theta_0}$. Let us consider the following function

(6) $$\varphi(z) = V^{\lambda_1}(z) + \ln\frac{1}{|1-z|} + 2g_{\theta_0}(z,\infty) - g_{\theta_0}(z,1).$$

φ is harmonic on $\bar{C} \setminus \Gamma_{\theta_0}$, and is constant on Γ_{θ_0} ([5, Теорема 4.4]) $\varphi(z) = \omega_1$. By maximum principle we have $\varphi(z) = \omega_1$, $z \in \bar{C}$.

When $z \to \infty$ we have

$$\omega_1 = 2\gamma_{\theta_0} - 2g_{\theta_0}(\infty,1) = 2\gamma_{\theta_0} - g_{\theta_0}(1,\infty)$$

where γ_{θ_0} is Rabben's constant of Γ_θ.

Function

$$\omega(z) = V^{\lambda_1}(z) + \ln\frac{1}{|1-z|} - g_\theta(z,1) + 2g_\theta(z,\infty),$$

is superharmonic on $\bar{C} \setminus \Gamma_\theta$.

Again, by maximum principle we have

$$\omega(\infty) \ge \min_{z\in\Gamma_\theta} \omega(z).$$

Further

$$\min_{z\in\Gamma_\theta}\omega(z) = \min_{z\in\Gamma_\theta}\left(V^{\lambda_1}(z) + \ln\frac{1}{|1-z|}\right) \ge \min_{|z|=1}\left(V^{\lambda_1}(z) + \ln\frac{1}{|1-z|}\right) =$$
$$\omega_1 = 2\gamma_{\theta_0} - g_{\theta_0}(1,\infty).$$

Since $\omega(\infty) = 2\gamma_\theta - g_\theta(\infty,1) = 2\gamma_\theta - g_\theta(1,\infty)$, we have

$$2\gamma_\theta - g_\theta(1,\infty) \ge 2\gamma_{\theta_0} - g_{\theta_0}(1,\infty),$$

and

$$\frac{d}{d\theta}\left[2\gamma_\theta - g_\theta(1,\infty)\right] = 0.$$

Let $F(z,\theta_0)$ be a conformal mapping from $\bar{C} \setminus \Gamma_{\theta_0}$ on $\{z : |z| > 1\}$, such that $F(\infty,\theta_0) = \infty$. It is easy to see that

$$F(z,\theta_0) = \frac{z+1+\sqrt{z^2-2z\cos\theta_0+1}}{2\cos\frac{\theta_0}{2}},$$

and so

$$g_{\theta_0}(z,\infty) = \ln\left|\frac{z+1+\sqrt{z^2-2z\cos\theta_0+1}}{2\cos\frac{\theta_0}{2}}\right| \tag{7}$$

and

$$g_{\theta_0}(z,1) = \ln\left|\frac{1-F(1,\theta_0)F(z,\theta_0)}{F(z,\theta_0)-F(1,\theta_0)}\right|. \tag{8}$$

From (7) and (8) we have

$$\gamma_\theta = \ln\frac{1}{\cos\frac{\theta}{2}},$$

and

$$2\gamma_\theta - g_\theta(1,\infty) = \ln\frac{1}{\cos\frac{\theta}{2}(1+\sin\frac{\theta}{2})}. \tag{9}$$

From (9), we see that θ_0 is solution of the equation

$$2\sin^2\frac{\theta}{2} + \sin\frac{\theta}{2} - 1 = 0.$$

Hence

$$\theta_0 = \frac{\pi}{3} \quad \text{and} \quad \omega_1 = \ln\frac{4}{3\sqrt{3}}.$$

From (6), (7) and (8), we have

$$V^{\lambda_1}(z) + \ln\frac{1}{|1-z|} - \omega_1 = \ln\left|\frac{(z+1+\sqrt{z^2-z+1})(z-2+\sqrt{z^2-z+1})^2}{3\sqrt{3}(z+\sqrt{z^2-z+1})^2}\right| \tag{10}$$

Proof of Gonchar–Rahmanov Theorem:

Let $\pi_n = \pi_n(f) = \frac{p_n}{q_n}$, $\deg p_n = n$, $\deg q_n = n$. From (2) we have that $\frac{(q_n f - p_n)(z)}{z^{2n+1}} \in H(U)$. By the Caushy integral formula it follows that

$$Q(z)\frac{(q_n f - p_n)(z)}{z^{2n+1}} = \frac{1}{2\pi i}\int_{\partial U} \frac{(q_n f - p_n)(t)}{t^{2n+1}}\,\frac{1}{t-z}\,dt, \; z \in U, \quad \text{where } Q(z) \in P_n.$$

So we have

$$(11) \qquad (f - \pi_n)(z) = \frac{1}{2\pi i} \int\limits_{\partial U} \frac{z^{2n+1}}{t^{2n+1}} \frac{(Qq_n)(t)}{(Qq_n)(z)} \frac{f(t)\,dt}{t - z} .$$

Now, for $||f||_{\partial U} = M$, $\rho(z, \partial U) = \min\limits_{t \in \partial U} |t - z|$, we see that

$$(12) \qquad |(f - \pi_n)(z)| \leq \frac{M}{\rho(z, \partial U)} |z|^{2n+1} \frac{||q_n Q||_{\partial U}}{|(q_n Q)(z)|}.$$

Let ν_n and μ_n are measures associated, respectively, with polynomials Q and q_n; (i.e., if $P_n(z) = \prod\limits_{k=1}^{n} (z - z_k)$, then measure associated with polynomial P_n, μ_n, is given by $\mu_n = \frac{1}{n} \sum\limits_{k=1}^{n} \delta(z_k)$, where $\delta(z_k)$ is the Dirac measure concentrated at z_k, $k = 1, 2, \ldots, n$). From (12) we have

$$(13) \quad \frac{1}{n} \ln |(f - \pi_n)(z)| \leq 2 \ln |z| + \left(V^{\nu_n} + V^{\mu_n}\right)(z) - \min_{t \in \partial U} \left(V^{\nu_n} + V^{\mu_n}\right)(t) + o(1) ,$$

Let Q be such that $s(\nu_n) \subset \partial U$ and $s(\mu_n) \subset C \setminus U$, $n \in \Lambda \subset N$.

Let μ_n' be measure associated with μ_n on ∂U_1 as in ([4, 5]), i. e. $(V^{\mu_n'} - V^{\mu_n}(z) = const$, $z \in U_1$; from (13) it follows

$$(14) \qquad \begin{aligned} &\frac{1}{n} \ln |(f - \pi_n)(z)| \leq \\ &2 \ln |z| + \left(V^{\nu_n} + V^{\mu_n'}\right)(z) - \min_{t \in \partial U} \left(V^{\nu_n} + V^{\mu_n'}\right)(t) + o(1), \\ &n \to \infty, \ n \in \Lambda, \ z \in U. \end{aligned}$$

Let Λ_1 be a subsequence of Λ, such that

$$\overline{\lim_{n \in \Lambda}} \frac{1}{n} \ln |(f - \pi_n)(z)| = \lim_{n \in \Lambda_1(z)} \frac{1}{n} \ln |(f - \pi_n)(z)|.$$

Further, let Λ_2 be a subsequence of Λ_1, such that $\mu_n' \to \mu$ and $\nu_n \to \nu$, where μ and ν are unit measures on ∂U. From [4, 5], we have

$$(15) \qquad \begin{aligned} &\lim_{n \in \Lambda_2(z)} \frac{1}{n} \ln |(f - \pi_n)(z)| \leq \\ &2 \ln |z| + \left(V^{\nu_n} + V^{\mu_n}\right)(z) - \min_{t \in \partial U} \left(V^{\nu_n} + V^{\mu_n}\right)(t), \quad z \in U. \end{aligned}$$

Set $\tilde{\mu}(\tilde{e}) = \mu(e)$, where $\tilde{e} = \frac{1}{e}$, and e is a Borel subset of ∂U.
For $z \in U$

$$(16) \qquad \begin{aligned} &2 \ln |z| + \left(V^{\nu} + V^{\mu}\right)(z) - \min_{t \in \partial U} \left(V^{\nu} + V^{\mu}\right)(t) = \\ &\left(V^{\tilde{\nu}} + V^{\tilde{\mu}}\right)\left(\frac{1}{z}\right) - \min_{t \in \partial U} \left(V^{\tilde{\nu}} + V^{\tilde{\mu}}\right)\left(\frac{1}{t}\right). \end{aligned}$$

Set $\tilde{\nu} = \int \lambda_t \, d\tilde{\mu}(t)$, where λ_t is measure on ∂U as in [5].
For $z \in U$, we have

$$(V^{\tilde{\nu}} + V^{\tilde{\mu}})\,(\frac{1}{z}) = \int (V^{\lambda_t} + V^{\delta_t})\,(\frac{1}{z})\,d\tilde{\mu}(t), \tag{17}$$

and

$$\begin{aligned}\min_{|z|=1}(V^{\tilde{\nu}} + V^{\tilde{\mu}})\,(\frac{1}{z}) = \min_{|z|=1}\int (V^{\lambda_t} + V^{\delta_t})\,(\frac{1}{z})\,d\tilde{\mu}(t) \geq \\ \int \min_{|z|=1}(V^{\lambda_t} + V^{\delta_t})\,(\frac{1}{z})\,d\tilde{\mu}(t).\end{aligned} \tag{18}$$

From (17) and (18) we have

$$\begin{aligned}&(V^{\tilde{\nu}} + V^{\tilde{\mu}})\,(\frac{1}{z}) - \min_{|t|=1}(V^{\tilde{\nu}} + V^{\tilde{\mu}})\,(\frac{1}{t}) \leq \\ &\int \left[(V^{\lambda_t} + V^{\delta_t})\,(\frac{1}{z}) - \min_{|u|=1}(V^{\lambda_t} + V^{\delta_t})\,(\frac{1}{u})\right] d\tilde{\mu}(t) \leq \\ &(V^{\lambda_\xi} + V^{\delta_\xi})\,(\frac{1}{z}) - \min_{|t|=1}(V^{\lambda_\xi} + V^{\delta_\xi})\,(\frac{1}{t}), \quad |\xi| = 1, \quad \arg\xi = \arg\frac{1}{z}.\end{aligned} \tag{19}$$

From (16) and (19), we have

$$\overline{\lim_{n\in\Lambda}}\,\frac{1}{n}\ln|(f-\pi_n)(z)| \leq (V^{\lambda_\xi} + V^{\delta_\xi})\,(\frac{1}{z}) - \min_{|t|=1}(V^{\lambda_\xi} + V^{\delta_\xi})\,(\frac{1}{t}). \tag{20}$$

From (10) and (20) we have

$$\begin{aligned}&\overline{\lim_{n\in\Lambda}}\,\frac{1}{n}\ln|(f-\pi_n)(z)| \leq \\ &\ln\left|\frac{\left(\frac{\xi}{z}+1+\sqrt{(\frac{\xi}{z})^2-\frac{\xi}{z}+1}\right)\left(\frac{\xi}{z}-2+\sqrt{(\frac{\xi}{z})^2-\frac{\xi}{z}+1}\right)^2}{3\sqrt{3}\left(\frac{\xi}{z}+\sqrt{(\frac{\xi}{z})^2-\frac{\xi}{z}+1}\right)^2}\right|.\end{aligned} \tag{21}$$

It is easy to see that the right hand side in (21) is < 0, for $|z| < r$, where r is defined by the equation

$$2g_{\frac{\pi}{3}}(r,0) - g_{\frac{\pi}{3}}(r,1) = 0\,.$$

The proof is complete.

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