Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 6 (1992), 27–30

FILOMAT–20, Niš, September 26–28, 1991

COSINE OPERATOR FUNCTION CLASS V

MOMČILO MANDIĆ

ABSTRACT. *Basic concepts of operator function class* **V** *have been dealt with in this paper. The main result is given by theorem* 2.1 *stating the operator* **A** *generates the cosine operator function class* **V**. *The asymptotic resolvent has been used for this.*

0. Introduction. It is a well known fact that the theory of operator semigroups, which has been thoroughly studied, is related to Cauchy's abstract problem of the first order:

$$\frac{du}{dt} = Au(t)\,,\ t \geq 0\,; \quad u(0) = x \in D(A)\,, \tag{0.1}$$

where A is the assigned linear operator.

Similarly, the cosine operator function, and this is the family of bounded linear operators $\{C(t) : t \in R = (-\infty, \infty)\}$ acting from a vector space X into itself and satisfying D'Alembert's functional equation:

$$C(t+s) + C(t-s) = 2C(t)C(s), \quad t, s \in R \quad \text{and} \quad C(0) = I\,, \tag{0.2}$$

is closely related to Cauchy's abstract problem of the second order:

$$\frac{d^2u}{dt^2} = Au(t), \quad u(0) = x \in D(A), \quad u'(0) = 0\,. \tag{0.3}$$

Actually, the fact that $C(t)$ is the cosine function with infinitesimal generator A is equivalent, under some conditions, to the fact that $u(t) = C(t)x$, $x \in D(A)$, is a unique solution of equation (0.3). Analogously, as the theory of semigroups, there occurs a very important problem of how to generate the cosine operator function, based on the conditions which are satisfied by the assigned operator A. Both, in Banach's and in locally convex spaces, for the case when $C(t)$ is an equicontinuous function, the below mentioned statement plays the main role:

If $C(t)$ is an equicontinuous cosine function, than its Laplace's transformation exists, and for its infinitesimal generator A there is a bounded generator $R(\lambda^2, A) = (\lambda^2 I - A)^{-1}$ such that the relation:

$$\lambda R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} C(t)x\,dt\,, \quad x \in X \tag{0.4}$$

is valid.

But, should it be assumed that, instead of the equal limitation, $C(t)$ is locally equicontinuous, e.g. at one segment $[-a, a]$, than the relation (0.4) in its general case is not valid, because it may happen that there is no Laplace's transformation or that there is no resolvent to operator A.

This work has, actually, been the study of locally equicontinuous cosine function, namely, in its general form, the so called class V, where V is the assigned linear operator. The basic method has been based on the concept of asymptotic resolvent. The asymptotic resolvent of a closed linear operator A implies a family of bounded linear operators $\{ Ra(\lambda) \, : \, \lambda > \omega, a > 0$ and $\omega > 0$ are constant $\}$ which are acting from a space X into X and satisfy certain conditions.

1. Basic Concepts of Cosine Operator function Class V. Let X be locally convex and sequentially complete linear topological space. Let $V \in L(X)$, where $L(X)$ is a set of all the linear continuous operators from X into X, for which an inverse V^{-1} operator exists, so that the $R(V)$ range operator V is a dense set all over in X.

DEF 1.1. *The operator function $C(t) \in L(X)$, $t \in R = (-\infty, \infty)$ is called a Class V Cosine Operator Function, if $C(t)$ satisfies the following conditions:*

(1.1) $(C(t+s) + C(t-s))V = 2C(t)C(s), \quad \forall t, s \in R$

(1.2) $C(0) = V$

(1.3) $s - \lim_{t \to s} C(t)x = C(s)x, \quad \forall s \in R$ *and* $\forall x \in X$.

Let us designate with $\{p\}$ the set of all the continuous seminorms defined on X.

DEF 1.2. *It is said for the cosine function $C(t)$ that it is $p \in \{p\}$ there is a continuous seminorm $q \in \{p\}$, such that for $\forall t \in K$, where $K \subset R$ is an arbitrary compact set, end for each $x \in X$, the following is true*

$$p(C(t)x) \le q(x).$$

Let us introduce $V^{-1}C(t)$, where $t \in R$ and $D(V^{-1}C(t)) = \{ x \in X \, : \, C(t)x \in R(V) \}$.

Operator $V^{-1}C(t)$ is closed and the following is true for it

$$R(V) \subseteq D(V^{-1}C(t)).$$

Now it is possible to define the operator A as follows:

$$D(A) = \left\{ x \in R(V) \, : \, \lim_{h \to 0} \frac{V^{-1}C(h)x - x}{h^2} \quad \text{exists} \right\}$$

and

$$Ax = lim_{h \to 0} \frac{2(V^{-1}C(h)x - x)}{h^2} \quad \text{for} \quad x \in D(A).$$

Operator A is closable. Operator $\hat{A}$, presenting the closure of operator A is called V-complete infinitesimal generator of $C(t)$ (V-c. i. g.).

2. The Main Results. Let operator V be from $L(X)$. For the closed linear operator A let us introduce the concept of asymptotic resolvent class V.

DEF 2.1. *Let $a > 0$ be a fixed number. The family of operators $Ra(\lambda) \in L(x)$, $\lambda.\omega > 0$, ω - a constant, is called the asymptotic resolvent class V, of the closed linear operator A with the domain $D(A)$, if it satisfies the following conditions:*

(2.1) *For $\forall x \in X$, $Ra(\lambda)$ is infinitely differentiable of λ for each $\lambda > \omega$ and $Ra(\lambda)$ maps X into $D(A)$.*

(2.2) $Ra(\lambda)Ra(\mu)x = Ra(\mu)Ra(\lambda)x$ *for* $\forall x \in X$ *and for* $\forall\lambda > \omega$ *and* $\mu > \omega$.

(2.3) $ARa(\lambda)x = Ra(\lambda)Ax$ *for* $\forall x \in D(A)$ *and* $\lambda > \omega$.

(2.4) $Ra(\lambda)Vx = VRa(\lambda)x$ *for* $\forall x \in X$.

(2.5) $(\lambda^2 I - A)Ra(\lambda)x = Vx + U(\lambda)x$ *for* $\forall x \in X$, *where operators* $U(\lambda)$ *belong to* $L(X)$ *and satisfy the following condition:*
For each $x \in X$, $U(\lambda)x$ is an infinitely differentiable function of λ and for each continuous seminorm p, there is a continuous seminorm q, so that the following is true:

(2.6) $p[[\frac{e^{a\lambda}U(\lambda)}{\lambda}]x]^{(n)} \leq \frac{n!}{\lambda^{n+1}}q(x)$ *for each* $x \in X$, $\lambda > \omega$ *and* $n = 1, 2, 3, \ldots$

DEF 2.2. *For two asymptotic resolvents $Rb(\lambda)$ and $Ra(\lambda)$, $b > a > 0$, it is said that they are compatible if in the following relation:*

(2.7) $Rb(\lambda)x = Ra(\lambda)x + \Phi(\lambda)x$, $x \in X$ *where $\Phi(\lambda)$ is an infinitely differentiable function by λ , $\lambda > \omega$ and satisfies the following requirement:*
For each continuous seminorm $p \in \{p\}$ there is a continuous seminorm $q \in \{p\}$ which is such that the following is true:

(2.8) $p(\lambda\Phi(\lambda)x)^{(n)} \leq (a^{n+1}e^{-a\lambda} + b^{n+1}e^{-b\lambda} + \phi(\lambda, n))q(x)$ *and* $\phi(\lambda, n) \approx \frac{a^n e^{-a\lambda}}{\lambda}$ *when* $\lambda \to \infty$ *and* $n = 1, 2, 3 \ldots$

The main result of this work is given by the following theorem:

THEOREM 2.1. *Let X be a sequentially complete and locally convex topological space. For the linear operator A to be V-c. i. g. locally semicontinuous cosine operator function class V, defined in unique manner, it is necessary and sufficient for it to satisfy the following conditions:*

(2.9) *A is a closed linear operator with an all over dense domain $D(A)$.*

(2.10) $AVx = VAx$ *for* $\forall x \in D(A)$.

(2.11) *The $VD(A)$ set is the core of the operator A.*

(2.12) *For each $a > 0$ there is an asymptotic resolvent $Ra(\lambda)$ class V, of operator A, satisfying the condition:*
For each continuous seminorm $p \in \{p\}$, there is a continuous seminorm $q \in \{p\}$ such that the following is true:

(2.13) $\quad p((\lambda R(\lambda))^{(n)}x) \leq \left(a^{n+1}e^{-a\lambda} + \frac{n!}{\lambda^{n+1}}\right)q(x)$ *for each* $x \in X$, $\lambda > \omega$ *and* $n = 1, 2, 3, \ldots$

(2.14) *For* $b > a > 0$, *the asymptotic resolvents* $Rb(\lambda)$ *and* $Ra(\lambda)$ *are compatible.*

REFERENCES

[1] S. OUCHI, *Semi-groups of operators in locally convex spaces*, J. Math. Soc. Japan, 25(2), (1973), 265–276.

[2] F. VAJZOVIĆ, *On cosine operator functions in Banach spaces*, Akad. N. i umj., 59, (1982), 23–30.

[3] T. KOMURA, *Semigroups of Operators in locally convex spaces*, J. Funct. Anal., 2, (1968), 258–296.

[4] T. KOMURA, *Semigroups of operators in locally convex spaces*, Proc. Int. Conf. Funct. Akal. and Relat. Topics, Tokyo 1969, Tokyo, (1970), 353–360.

[5] Y. KONISHI, *Cosine functions of operators in locally convex spaces*, J. FAC. Sci. Univ. Tokyo, Sec 1A, 18(3), (1972), 443–463.

[6] M. SOVA, *Cosine operator functions*, Rozprawy matematyczne , 49, Warszawa (1966), 3–46.

[7] E. HILLE, R. FILLIPS, *Funkcionaljnij analiz i polugrupi*, Moskva, 1962.

[8] K. IOSIDA, *Funkcionaljnij analiz*, Mir, Moskva, 1967.

[9] S. KUREPA, *Funkcionalna analiza*, Školska knjiga, Zagreb 1981.

[10] J. KISYNSKI, *On cosine operator functions and one-parameter groups of operators*, Studia mathematica, 44, (1972), 93–105.

[11] N. TANAKA, *On the Exponentially Bounded C-semigroups*, J. Math., 10(1), (1987), 107–117.

Saobraćajni fakultet Sarajevo
71 320 Vogošća, Igmanskog marša 21
Yugoslavia