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SOME THEOREMS ON A LOCAL (NONASSOCIATIVE) NEAR-RING

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ABSTRACT. In this paper we investigate some relations among a local (non-associative) ring S and its subsets $L, L_d, S \setminus L$ and A, and their associators in S (where $L(L_d)$ is the set of all elements from S which do not have a left (right) inverse in S and A the set of all elements which do not have neither left nor right inverses).

DEFINITION 1. A unitary right (nonassociative) near-ring is a nonempty set S with two binary operations: addition (+) and multiplication (·), such that:

- 1. The elements of S form a group (S, +) under addition,
- 2. The elements of S form a groupoid (S, \cdot) ,
- 3. $\forall x \in S, x \cdot 0 = 0$, where 0 is the additive identity of S,
- 4. There exists an element $1 \in S$ such that $1 \cdot s = s \cdot 1 = s$, for all $s \in S$
- 5. $\forall x, y, z \in S, (x+y) \cdot z = x \cdot z + y \cdot z.$

Let L be the subset of S of all elements without left inverses, i.e. $L = \{l \in S \mid S \neq Sl\}$.

DEFINITION 2. S is said to be a local near-ring if L is a left S-subgroup (Df. 2.1.[2]).

Denote set $S \setminus L$ by U, the set of all elements of S without a right inverse by L_d ; the associators: $\{(xy)z - x(yz) \mid x, y, z \in S\}$ of S by A(S), $\{(xs)l - x(sl) \mid l \in L, s, x \in S\}$ of L by $A_{rl}(L)$, $\{(sl)x - s(lx) \mid l \in L, s, x \in S\}$ of L by $A_{ir}(L)$, $\{(ls)x - l(sx) \mid l \in L, s, x \in S\}$ of L by $A_{rr}(L)$, $\{(as)x - a(sx) \mid a \in A, s, x \in S\}$ of L by $L_{rr}(L)$, $L_{rr}(L)$,

LEMMA 1. Let S be a local near-ring. If $A_r(U)$, $A_{ir}(U) \subseteq L$, then the elements of L do not have right inverses, (See L. 1[1]).

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PROOF. Suppose that there exists some element l of L with a right inverse l'. Then $ll'=1\in U$ contradicts $ll'\in L$. Hence, the elements of L do not have right inverse in L. Let some element $l\in L$ have a right inverse $u\in U$, i.e. lu=1. Since $ul\in L$ then $1-ul\in U$ and there exists $x\in S$ such that x(1-ul)=1. As (1-ul)u=u-(ul)u=-a, then u=(x(1-ul))u and, from here, $(x(1-ul))u=a_1+x((1-ul)u)=a_1+x(u-(ul)u)=a_1+x(-a)\in L$, where a,a_1 are the associators of the ordered triples (u,l,u) and (x,(1-ul),u) respectively. This contradiction establishes the lemma.

Theorem 1. Let S be a unitary right (nonassociative) near-ring, and L be a subgroup of (S, +). If $A_{rl}(L) = \{0\}$, then S is local. Conversely, if S is a unitary local near-ring then $A_{rl}(L) \subseteq L$.

PROOF. If S is a unitary near-ring, L is a subgroup and $A_{rl}(L) = \{0\}$, then S is local. Otherwise, if $sl \notin L$, for some $l \in L$ and some $s \in S$, then x(sl) = 1, for some $x \in S$. Since, (xs)l - x(sl) = 0, for all $l \in L$ and all $s, x \in S$ and since x(sl) = 1, for some $l \in L$ and some $s, x \in S$, then (xs)l - 1 = 0 and, from here (xs)l = 1, for some $l \in L$ and some $x, s \in S$. This is a contradiction (because $l \in L$). Thus, $sl \in L$, for all $l \in L$ and all $s \in S$, i.e. L is a left S-subgroup and S is, by definition, a local near-ring.

Conversely, if S is a local near-ring, i.e. if $sl \in L$, for all $l \in L$ and all $s \in S$, then $(xs)l - x(sl) \in L$, for all $s, x \in S$ and all $l \in L$, i.e. $A_{rl}(L) \subseteq L$.

Theorem 2. Let S be a unitary right (nonassociative) near-ring. If L and $U \cup \{0\}$ are subgroups of (S, +) and $A_{rl}(L) \subseteq L$, then S is local.

PROOF. If S is a unitary near-ring, L is a subgroup of (S,+) and $A_{rl}(L) \subseteq L$, then S is local. Otherwise, if $sl \notin L$, for some $l \in L$ and some $s \in S$, then x(sl) = 1, for some $x \in S$. Since, $(xs)l - x(sl) = a \in L$, for all $l \in L$ and all $x, s \in S$, and since x(sl) = 1, for some $l \in L$ and some $s, x \in S$, then (xs)l - 1 = a, for some $l \in L$ and some $s, x \in S$. Since $a \in L$, then $(xs)l = a + 1 \in U$, i.e. $(xs)l \in U$ and $(xs)l - 1 \in (U \cup \{0\})$, because $(U \cup \{0\}, +)$ is a group. Thus, (xs)l - 1 = 0 and, from here (xs)l = 1, for some $l \in L$ and some $y = xs \in S$. This is a contradiction. Thus, $sl \in L$, for all $l \in L$ and all $s \in S$. So, S is a local near-ring.

THEOREM 3. Let S be a right unitary near-ring, $A_{ir}(L) = \{0\}$, $A_r(U) \subseteq L$ and every element of U has a right inverse. Then, S is a local near-ring if and only if L is a right S-subgroup.

PROOF. 1° If S is a right unitary near-ring, $A_{ir}(L) = \{0\}$ and L is a right S-subgroup, then S is a local near-ring. Otherwise, if $sl \notin L$, for some $l \in L$ and some $s \in S$, then (sl)x = 1, for some $x \in S$. Since, (sl)x - s(lx) = 0, for all $l \in L$ and all $x, s \in S$, and since (sl)x = 1, for some $l \in L$ and some $s, x \in S$, then 1 - s(lx) = 0 and, from here, s(lx) = 1, for some $s, x \in S$ and some $l \in L$. Since $k = lx \in L$ (by the assumption of this theorem), then sk = 1 is a contradiction. So, $sl \in L$, for all $l \in L$ and all $s \in S$, i. e. L is a left S-subgroup. Thus, S is a local near-ring.

 2^o If S is a right local near-ring, $A_{ir}(L) = \{0\}$ and $A_r(U) \subseteq L$, then L is a right S-subgroup. Otherwise, if $ls \notin L$, for some $l \in L$ and some $s \in S$, then

x(ls)=1, for some $x\in S$. Since (xl)s-x(ls)=0, for all $s,x\in S$ and all $l\in L$, and since x(ls)=1, for some $l\in L$ and some $s,x\in S$, then (xl)s-1=0 and, from here, (xl)s=1, for some $l\in L$ and some $s,x\in S$. Since $k=xl\in L$, then ks=1 contradicts L. 1. Thus, $ls\in L$, for all $l\in L$ and all $s\in S$, i. e. S is a right S-subgroup.

THEOREM 4. Let S be a right unitary local near-ring, (L,+) and $(U \cup \{0\},+)$ be the subgroups of (S,+) and $A_r(U) \subseteq L$. Then, L is a right S-subgroup if and only if $A_{ir}(L) \subseteq L$.

PROOF. 1° If S is a unitary local near-ring, $A_{ir}(L) \subseteq L$, L and $(U \cup \{0\}, +)$ are the subgroups of (S, +), then L is a right S-subgroup. Otherwise, if $ls \notin L$, for some $l \in L$ and some $s \in S$, then x(ls) = 1, for some $x \in S$. Since $(xl)s - x(ls) = a \in L$, for all $s, x \in S$ and all $l \in L$, and since x(ls) = 1, for some $s, x \in S$ and some $l \in L$, then (xl)s - 1 = a. Since $a \in L$, then $(xl)s = a + 1 \in U$, i.e. $(xl)s \in U$ and, from here $a = (xl)s - 1 \in (U \cup \{0\})$, because $(U \cup \{0\}, +)$ is a group. Accordingly, a = (xl)s - 1 = 0 and, from here, (xl)s = 1, for some $l \in L$ and some $s, x \in S$. Since $k = xl \in L$ and $s \in S$, then (xl)s = 1 is a contradiction to L. 1. (because $A_{ir}(U) \subseteq A_{ir}(L)$). Thus, $ls \in L$, for all $l \in L$ and all $s \in S$, i.e. L is a right S-subgroup.

2° Conversely, if S is a right local near-ring and if L is a right S-subgroup, then $A_{ir}(L) \subseteq L$. Really, since $sl, ls \in L$, for all $l \in L$ and all $s \in S$, then $(sl)x - s(lx) \in L$, for all $l \in L$ and all $s, x \in S$. Thus, $A_{ir}(L) \subseteq L$.

THEOREM 5. Let S be a right unitary local near-ring. If $A_r(U)$, $A_{ir}(U) \subseteq L$, $A_{rr}(L) = \{0\}$ and L_d is a left S-subgroup, then $L_d = L$, L and L_d are the right S-subgroups. Conversely, if L_d is a right S-subgroup then $A_{rr}(L)$, $A_{ir}(L) \subseteq L$.

PROOF. If S is local, $A_r(\mathbb{U})$, $A_{ir}(\mathbb{U}) \subseteq L$, L_d is a left S-subgroup and $A_{rr}(\mathbb{L}) = \{0\}$, then $L_d = L$, L and L_d are right S-subgroups. Otherwise, if $ls \notin L$, for some $l \in L$ and some $s \in S$, then (ls)x = 1, for some $x \in S$. From (ls)x - l(sx) = 0, for all $l \in L$ and all $s, x \in S$ and from (ls)x = 1, for some $l \in L$ and some $s, x \in S$, follows 1 - l(sx) = 0 and l(sx) = 1, for some $l \in L$ and some $s, x \in S$. This is a contradiction to L.1. So, $ls \in L$, for all $l \in S$ and all $s \in S$, i.e. L is a right S-subgroup. Since L is a maximal S-subgroup (Th. 2.[1]) and L_d is a left S-subgroup, then $L_d \subseteq L$. Since the elements of L do not have right inverses (L. 1.), then $L \subseteq L_d$. Thus, $L_d = L$ and L_d is a right S-subgroup.

Conversely, if S is a right local near-ring and L_d is a right S-subgroup then $A_{rr}(L) \subseteq L$. Really, if $sl, ls \in L$, for all $l \in L$ and all $s \in S$, then $(ls)x - l(sx) = a \in L$ and $(sl)x - s(lx) \in L$, for all $l \in L$ and all $s, x \in S$. This means, $A_{rr}(L)$, $A_{ir}(L) \subseteq L$.

THEOREM 6. Let S be a right unitary local near-ring, $A_r(U)$, $A_{ir}(U) \subseteq L$, $(U \cup \{0\}, +)$ be a subgroup of (S, +) and L_d be a left S-subgroup. Then, $A_{rr}(L) \subseteq L$ if and only if $L_d = L$ and L_d is a right S-subgroup.

PROOF. 1° If L and L_d are left S-subgroups, $A_r(U)$, $A_{ir}(U)$, $A_{rr}(L) \subseteq L$ and $(U \cup \{0\}, +)$ is a group, then L_d is a right S-subgroup. Otherwise, if $ls \notin L$, for some

 $l \in L$ and some $s \in S$, then (ls)x = 1, for some $x \in S$. From $(ls)x - l(sx) = a \in L$, for all $l \in L$ and all $s, x \in S$, and from (ls)x = 1, for some $l \in L$ and some $s, x \in S$, follows 1 - l(sx) = a. Since $a \in L$, then $l(sx) = -a + 1 \in U$, i.e. $l(xs) \in U$. So, $a = 1 - l(sx) \in (U \cup \{0\})$, because $(U \cup \{0\}, +)$ is a group. From here, l(sx) = 1, for some $l \in L$ and some $s, x \in S$. This is a contradiction to L.1. So, $ls \in L$, for all $l \in L$ and all $s \in S$, i.e. L is a right S-subgroup. Since L is a maximal left S-subgroup (Th. 2. [1]) and L_d (by the assumption of this theorem) is a left S-subgroup then $L_d \subseteq L$. Since the elements of L do not have right inverses (L.1.) then $L \subseteq L_d$. Thus, $L_d = L$ and L_d is a right S-subgroup.

2° If S is a unitary right local near-ring, $L_d = L$ and L_d is a right S-subgroup then $A_{rr}(L) \subseteq L$. Really, since $L_d = L$ and $ls \in L_d$, for all $l \in L$ and all $s \in S$, then $ls \in L$ and $(ls)x - l(sx) \in L$, for all $l \in L$ and all $s, x \in S$. So, $A_{rr}(L) \subseteq L$.

THEOREM 7. Let S be a unitary right (nonassociative) near-ring, A and $U \cup \{0\}$ be the subgroups of (S, +), $L_d \subseteq L$ and $A_{rl}(A) \subseteq A$, then A is a left S-subgroup and S is local. Moreover, if A is a right S-subgroup, then $A_{rr}(A) = A_{rr}(L)$, $A_{ir}(A) = A_{ir}(L)$ and $A_{rr}(L_d)$, $A_{ir}(L_d)$, $A_{ir}(A)$, $A_{rr}(A)$, $A_{rr}(L)$, $A_{ir}(L) \subseteq L$.

PROOF. 1° If S is a unitary, right near-ring, A and $U \cup \{0\}$ are subgroups of (S,+), $L_d \subseteq L$ and $A_{rl}(A) \subseteq A$ then (by definitions of L_d , L and A and by the assumption, L = A, $A \supseteq L_d$), A and L are left S-subgroups. Otherwise, if $sa \notin A$, for some $a \in A$ and for some $s \in S$, then x(sa) = 1, for some $x \in S$. Since, by the assumption, $(xs)a - x(sa) = a' \in A$, for all $a \in A$ and all $s, x \in S$, and since x(sa) = 1, for some $a \in A$ and some $s, x \in S$, then $(xs)a - 1 = a' \in A$, for some $a \in A$ and some $s, x \in S$. Since L = A, then $a, a' \in L$. Since $a' \in A$ then $(xs)a = a' + 1 \in U$, i.e. $(xs)a \in U$ and, from here, $a' = (xs)a - 1 \in (U \cup \{0\})$, because $U \cup \{0\}$ is a group, i.e. $a' \in U \cup \{0\}$. From here, (xs)a - 1 = 0, i.e. (xs)a = 1, for some $a \in A$ and some $x, s \in S$. This is a contradiction. So, $sa \in A$, for all $a \in A$ and all $s \in S$, i.e. A and L are left S-subgroups. Thus, S is a local near-ring.

Moreover, if A is a right S-subgroup. then from $as, sa \in A$ follows: $\{(as)x - a(sx) \mid a \in A, s, x \in S\}$, $\{(sa)x - s(ax) \mid a \in A, s, x \in S\} \subseteq A$. Since L = A and $L_d \subseteq L = A$, then $A_{rr}(A)$, $A_{ir}(A)$, $A_{rr}(L)$, $A_{ir}(L)$, $A_{rr}(L_d)$, $A_{ir}(L_d) \subseteq L$.

Corollary 1. Let S be a unitary right near-ring, $U \cup \{0\}$ and A be the subgroups of (S, +), $L_d \subseteq L$ and $A_{rl}(L_d) \subseteq A$, then $A_{rl}(L_d)$, $A_{rl}(L) \subseteq L$ and $A_{rl}(L) = A_{rl}(A)$.

PROOF. Since L=A, $A_{rl}(L)=A_{rl}(A)$. Since $L_d\subseteq L$ and $A_{rl}(L)=A_{rl}(A)\subseteq L$, $A_{rl}(L_d)\subseteq A_{rl}(L)\subseteq L$, i.e. $A_{rl}(L_d)\subseteq L_d$.

Corollary 2. Let S be a unitary right near-ring, $U \cup \{0\}$ and A be the subgroups of (S, +), $L_d \subseteq L$ and $A_{rl}(A) \subseteq A$, then each element of U has a right inverse.

Theorem 8. Let S be unitary (nonassociative) near-ring, A be a subgroup of (S,+) and $L_d \subseteq L$. If $A_{rl}(A) = \{0\}$, then A is a left S-subgroup and S is local. Conversely, if A is a left S-subgroup or S is local then $A_{rl}(A) \subseteq L$.

Morcover, if A is a right S-subgroup, then $A_{ir}(A) = A_{ir}(L)$, $A_{rr}(A) = A_{rr}(L)$ and $A_{rr}(A)$, $A_{ir}(A)$, $A_{ir}(L)$, $A_{ir}(L)$, $A_{rr}(L)$, $A_{ir}(L)$, $A_{ir}(L)$ $\subseteq L$.

PROOF. If S is a unitary right near-ring, A is a subgroup of (S, +), $L_d \subseteq L$ and $A_{rl}(A) = \{0\}$, then A and L are left S-subgroups. Otherwise, if $sa \notin A$, for some $a \in A$ and some $s \in S$, then x(sa) = 1, for some $x \in S$. Since (xs)a - x(sa) = 0, for all $x, s \in S$ and all $a \in A$, and since x(sa) = 1, for some $x, s \in S$ and some $a \in A$, then (xs)a - 1 = 0 and, from here, (xs)a = 1, for some $a \in A$ and some $x, s \in S$.

Since L = A, by definitions of L, A and L_d and by the assumption, $a \in L$. Since $t = xs \in S$ then (xs)a = 1 is a contradiction. It means, $sa \in A$, for all $a \in A$ and all $s \in S$, i.e. A and L are left S-subgroups. Thus, S is a local near-ring.

Conversely, if S is a unitary local near-ring or if A is a left S-subgroup and $L_d \subseteq L$, then $A_{rl}(A) \subseteq L$. Really, since $sa \in A$, for all $a \in A$ and all $s \in S$, then $(xs)a - x(sa) \in A$, for all $a \in A$ and all $x, a \in S$. Since A = L then $A_{rl}(A) \subseteq L$.

Moreover, if A is a right S-subgroup, then from $sa, as \in A$, for all $a \in A$ and all $s \in S$, follows: $\{(as)x - a(sx) \mid a \in A, s, x \in S\}$, $\{(sa)x - s(ax) \mid a \in A, s, x \in S\} \subseteq A$. Since A = L and $L_d \subseteq L$, then $L_d \subseteq A$ and $A_{rr}(A)$, $A_{ir}(A)$, $A_{ir}(L)$, $A_{rr}(L)$, $A_{rr}(L)$, $A_{rr}(Ld)$, $A_{ir}(Ld) \subseteq L$.

COROLLARY 1. Let S be a unitary right near-ring, A be a subgroup of (S, +), $L_d \subseteq L$ and $A_{rl}(A) \subseteq \{0\}$, then $A_{rl}(L) = A_{rl}(L_d) = \{0\}$.

PROOF. Since L = A, $A_{rl}(L) = A_{rl}(A) = \{0\}$, i.e. $A_{rl}(L) = \{0\}$. Since $L_d \subseteq L$ and $A_{rl}(L) = \{0\}$, $A_{rl}(L_d) = \{0\}$.

COROLLARY 2. If S is a unitary right near-ring, A a subgroup of (S, +), $L_d \subseteq L$ and $A_{rl}(A) = \{0\}$, then each element of $S \setminus L$ has a right inverse.

If S is an associative unitary right near-ring then the condition $A_{rl}(A) = \{0\}$ is automatically fulfilled. Moreover, L is a left S-subgroup if and only if L is a subgroup of (S, +). If S is an associative local near-ring then the conditions: $A_{rl}(A) = \{0\}$ and $L_d = L = A$ are automatically fulfilled too (see L. 2. 4[2]).

COROLLARY 3. Let S be a unitary right (associative) near-ring, $L_d \subseteq L$ and A be a subgroup of (S, +). Then, A is a left S-subgroup and S is local, (See Th. 2. 5.[2]).

Theorem 9. Let S be unitary, right (nonassociative) near-ring, A be a subgroup of (S, +) and $L_d \supseteq L$. Then $A_{rr}(A) = \{0\}$ is a sufficient condition for L_d to be a right S-subgroup. Moreover, if L_d is a left S-subgroup, then S is local.

PROOF. If S is a unitary right near-ring, (A, +) is a group, $L_d \supseteq L$ and $A_{rr}(A) = \{0\}$, then A is a right S-subgroup and, therefore, L_d is a right S-subgroup. Otherwise, if $as \notin A$, for some $a \in A$ and some $s \in S$, then (as)x = 1, for some $x \in S$. Since (as)x - a(sx) = 0, for all $a \in A$ and all $s, x \in S$, and since (as)x = 1, for some $a \in A$ and some $s, x \in S$, then 1 - a(sx) = 0 and, from here, a(sx) = 1, for some $a \in A$ and some $s, x \in S$. Since $a \in A$, $y = sx \in S$ and since $A = L_d$ (by definitions of L_d and A and by the assumption), then a(sx) = 1 is a contradiction. It means, $as \in A$, for all $a \in A$ and all $s \in S$, i.e. A is a right S-subgroup. Since $L_d = A$, then L_d is a right S-subgroup. Since $A \supseteq L$ and $A_{rr}(A) = \{0\}$, $A_{rr}(L) = \{0\}$.

Moreover, if L_d is a left S-subgroup, then S is local. Really, in any case $L \subseteq A$. Since $A = L_d$ is a left S-subgroup then $A \subseteq L$. So, L = A, i.e. L is a left S-subgroup. Thus, S is a local near-ring.

COROLLARY 1. Let S be a unitary right near-ring, A be a left S-subgroup, $L_d \supseteq L$ and $A_{rr}(A) = \{0\}$, then each element of $S \setminus A$ has a right inverse.

COROLLARY 2. Let S be a unitary right near-ring, A be a left S-subgroup, $L_d \supseteq L$ and $A_{rr}(A) = \{0\}$, then $A_{rl}(L)$, $A_{ir}(L)$, $A_{rl}(A)$, $A_{ir}(A) \subseteq L$.

PROOF. Since S is, by Th. 9., local and L=A is a right S-subgroup, then from $sl, ls \in L$, for all $l \in L$ and all $s \in S$, follows $A_{rl}(L)$, $A_{ir}(L)$, $A_{rl}(A)$, $A_{ir}(A) \subseteq L$.

COROLLARY 3. Let S be a unitary right associative near-ring, A be a subgroup of (S,+). If $L_d \supseteq L$ then L_d is a right S-subgroup. Moreover, if A is a left S-subgroup, then S is local.

From the proof of Th. 9. follows the next theorem.

Theorem 10. Let S be a unitary, right (nonassociative) near-ring and A be a subgroup of (S, +). Then $A_{rr}(A) = \{0\}$ is a sufficient condition for A to be a right S-subgroup. Moreover, if A is a left S-subgroup, then S is local.

COROLLARY 1. Let S be a unitary right associative near-ring, and A be a subgroup of (S, +). Then, A is a right S-subgroup.

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