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$\mathfrak{L}_n$-SEMIGROUPS

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Abstract. *Е.Г.Шутов* [3] *and N.Kimura, T.Tamura and R.Merkel* [2] *considered* λ*-semigroups, i.e. semigroups in which every subsemigroup is a left ideal. S.Bogdanović and the author treated in* [1] λ_n*-semigroups, i.e. semigroups in which*

(1) $$S^n A = A^{n+1}$$

for every subsemigroup A *of* S. *In this paper we generalize the results from* [1]. *We prove that* S *is an* $\mathfrak{L}_n$*-semigroup if and only if* S *is a retractive extension of a right zero band by an* $\mathfrak{L}_n$*-nil-semigroup (Th.* 3, 4).

DEFINITION 1. *A semigroup* S *is an* $\mathcal{L}_n$*-semigroup if for every its subsemigroup* A *the following condition holds:*

(2) $$S^n A \subseteq A\,.$$

LEMMA 1. *A semigroup* S *is an* $\mathfrak{L}_n$*-semigroup if and only if*

(3) $$(\forall a \in S)\ S^n a \subseteq \langle a\rangle\ .$$

PROOF. Let S be an $\mathfrak{L}_n$-semigroup. Then
$$S^n a \subseteq S^n \langle a\rangle \subseteq \langle a\rangle$$
for all $a \in S$, i.e. (2) holds.

Conversely, let A be a subsemigroup of S. Then for $a \in A$ we have that $S^n a \subseteq \langle a\rangle$, and thus $S^n A \subseteq A$. Therefore, S is an $\mathfrak{L}_n$-semigroup. □

LEMMA 2. *Every* λ_n*-semigroup is an* $\mathcal{L}_n$*-semigroup.*

PROOF. Follows immediately. □

The converse of Lemma 2. is not true, for example, the semigroup $\langle x\rangle = \{x, x^2, x^3, x^4, x^5 = x^6\}$ is an $\mathfrak{L}_2$-semigroup. But it is not a λ_2-semigroup, since for $\langle x^2\rangle$ the condition (1) not holds.

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LEMMA 3. *Every subsemigroup and every homomorphic image of an $\mathfrak{L}_n$-semigroup is an $\mathfrak{L}_n$-semigroup.*

PROOF. Follows immediately. □

LEMMA 4. *Let S be an $\mathfrak{L}_n$-semigroup. Then*

(i) S is periodic;

(ii) $E(S)$ is a right zero band;

(iii) $E(S)$ is an ideal of S;

(iv) $Reg(S) = E(S)$.

PROOF. (*i*). Let $x \in S$. Then by hypothesis and by (2) we have that
$$x^{2n+1} = x^{n-1}x^x x^2 \in S^n x^2 \subseteq \langle x^2 \rangle \ .$$
Thus, S is periodic.

(*ii*). Let $e \in E(S)$. Then by (3) we have that $S^n e \subseteq \langle e \rangle = e$, whence $f^n e = fe = e$ for all $e, f \in E(S)$. Therefore, $E(S)$ is a right zero band.

(*iii*). Let $x \in S$, $e \in E(S)$. Then by (3) we have that $xe = xe^{n-1}e = e$. Hence, $E(S)$ is a left ideal of S. Since
$$(ex)^2 = exex = (e^{n-1}xe)x = ex \in E(S) \ ,$$
we then have that $E(S)$ is, also, a right ideal of S. Thus $E(S)$ is an ideal of S.

(*iv*). Let $a \in Reg(S)$. Then $a = axa$, for some $x \in S$, so by (*iii*) it follows that $a = (ax)a \in E(S) \cdot S \subseteq E(S)$. Thus $Reg(S) \subseteq E(S)$. The opposite inclusion also holds. Therefore, $Reg(S) = E(S)$. □

LEMMA 5. *Let S be an $\mathfrak{L}_n$-semigroup. Then*

(i) if $n = 2k-1$, then for every $x \in S$ it is
$$\langle x \rangle = \{x, x^2, \dots, x^m = x^{m+1}\}$$
for some m, $1 \le m \le 2k+1$.

(ii) if $n = 2k$, then for every $x \in S$ it is
$$\langle x \rangle = \{x, x^2, \dots, x^m = x^{m+1}\}$$
for some m, $1 \le m \le 2k+3$.

PROOF. By Lemma 4. (*i*) we have that S is periodic. By Lemma 4. (*iii*), $\langle x \rangle$ has a zero for every $x \in S$. Hence
$$\langle x \rangle = \{x, x^2, \dots, x^m = x^{m+1}\}$$
for some $m \in \mathbf{Z}^+$. We have the following two cases:

(*i*). If $n = 2k-1$ and $m > 2k+1$, then
$$\langle x \rangle^{2k-1} x^2 \nsubseteq \langle x^2 \rangle \ ,$$
since $x^{2k-1} \notin \langle x^2 \rangle = \{x^2, x^4, \dots, x^m = x^{m+1}\}$, which is a contradiction.

(*ii*). If $n = 2k$ and $m > 2k+3$, then
$$\langle x \rangle^{2k} x^2 \nsubseteq \langle x^2 \rangle \ ,$$
since $x^{2k+3} \notin \langle x^2 \rangle = \{x^2, x^4, \dots, x^m = x^{m+1}\}$, which is not possible. □

THEOREM 1. *Let S be a cyclic semigroup. Then the following conditions are equivalent:*

(*i*) *S is an $\mathfrak{L}_{2k}$-semigroup;*

(*ii*) *S is an $\mathfrak{L}_{2k+1}$-semigroup;*

(*iii*) *S is a $(2k+3)$-nilpotent semigroup.*

PROOF. *(i) $\Longrightarrow$ (iii). Let $\langle x \rangle$ be an $\mathcal{L}_{2k}$-semigroup. Then by Lemma 5. x^{2k+3} is a zero of $\langle x \rangle$. Thus, $\langle x \rangle^{2n+3} = 0$.*

(ii) $\Longrightarrow$ (iii). This implication is similar to the (i) $\Longrightarrow$ (iii).

(iii) $\Longrightarrow$ (i). Let $S = \langle x \rangle$ and $S^{2k+3} = 0$. Then $x^{2k+3} = 0$. It is easy to verify that $S^{2k}a \subseteq \langle a \rangle$ holds for all $a \in S$. Thus, S is an $\mathfrak{L}_{2k}$-semigroup.

(iii) $\Longrightarrow$ (ii). Similarly as (iii) $\Longrightarrow$ (i). □

THEOREM 2. *Let S be a cyclic semigroup. Then S is an $\mathfrak{L}_1$-semigroup if and only if $S^3 = 0$.*

PROOF. *Follows by Lemma 5.* □

THEOREM 3. *The following conditions on a semigroup S are equivalent:*

(*i*) *S is an $\mathfrak{L}_{2k}$-semigroup;*

(*ii*) *$(\forall x_1, x_2, \dots, x_{2k}, y \in S)\ x_1x_2\cdots x_{2k}y \in \{y^2, y^3, \dots, y^{2k+3}\}$;*

(*iii*) *S is a retractive extension of a right zero band by an $\mathfrak{L}_{2k}$-nil-semigroup.*

PROOF. (*i*) $\Longrightarrow$ (*ii*). Let S be an $\mathcal{L}_{2k}$-semigroup. Then by Lemmas 1. and 5. we have that

$$x_1x_2\cdots x_{2k}y \in \{y, y^2, \dots, y^{2k+3} = y^{2k+4}\}$$

for all $x_1, x_2, \dots, x_{2k}, y \in S$. If $x_1x_2\cdots x_{2k}y = y$, then

$$\begin{aligned}(x_1x_2\cdots x_{2k})^{2k+3}y &= (x_1x_2\cdots x_{2k})^{2k+2}(x_1x_2\cdots x_{2k})y \\ &= (x_1x_2\cdots x_{2k})^{2k+2}y = \cdots \\ &= (x_1x_2\cdots x_{2k})y \\ &= y \in \langle y \rangle \cap E(S)\ ,\end{aligned}$$

since, by Lemma 4. (*iii*),

$$(x_1x_2\cdots x_{2k})^{2k+3}y \in E(S) \cdot S \subseteq E(S)\ .$$

Hence, $y = y^{2k+3}$, and so the condition (*ii*) holds.

(*ii*) $\Longrightarrow$ (*i*). Let A be a subsemigroup of S. Then for every $y \in A$, $x_1x_2\cdots x_{2k}y \in \langle y \rangle \subseteq A$. Thus $S^{2k}A \subseteq A$, i.e. S is an $\mathfrak{L}_{2k}$-semigroup.

(*i*) $\Longrightarrow$ (*iii*). Let S be an $\mathfrak{L}_{2k}$-semigroup. Then by Lemma 4. $E(S)$ is a right zero band and it is an ideal of S. The mapping $\varphi : S \to E(S)$ defined by $\varphi(x) = x^{2k+3}$ is a retraction. Indeed,

$$\begin{aligned}\varphi(xy) &= (xy)^{2k+3} \\ &= ((xy)^{2k+2}x)y \\ &= y^{2k+3}\ , \qquad \text{since } ((xy)^{2k+2}x)y \in \langle y \rangle \cap E(S) \\ &= \varphi(y)\end{aligned}$$

$$= \varphi(x)\varphi(y) \ , \qquad \text{since } E(S) \text{ is a left zero band}$$

and

$$\varphi^2(x) = \varphi(x) \ .$$

It remains to prove that the Rees factor semigroup $V = S/E(S)$ is an $\mathfrak{L}_{2k}$-nil-semigroup. It is clear that V is an $\mathcal{L}_{2k}$-semigroup. For every $x \in S - E(S)$ there exists $n \in \mathbb{Z}^+$ such that $x^n \in E(S)$, so V is a nil-semigroup.

$(iii) \implies (i)$. Let A be a subsemigroup of S, let $x_1, x_2, \dots, x_{2k} \in S$ and let $y \in A$. Then

$$x_1 x_2 \cdots x_{2k} y \in E(S) \quad \text{or} \quad x_1 x_2 \cdots x_{2k} y \notin E(S) \ .$$

If $x_1 x_2 \cdots x_{2k} y \in E(S)$, then

$$\begin{aligned} x_1 x_2 \cdots x_{2k} y &= \varphi(x_1 x_2 \cdots x_{2k} y) \ , && \text{since } \varphi \text{ is a retraction} \\ &= \varphi(x_1)\varphi(x_2)\cdots\varphi(x_{2k})\varphi(y) \\ &= \varphi(y) \ , && \text{since } E(S) \text{ is a right zero band.} \end{aligned}$$

Since there exists $n \in \mathbb{Z}^+$ such that $y^n \in E(S)$, then $\varphi(y) = (\varphi(y))^n = \varphi(y^n) = y^n \in \langle y \rangle \subseteq A$. Therefore, in this case

$$x_1 x_2 \cdots x_{2k} y \in A \ .$$

If $x_1 x_2 \cdots x_{2k} y \notin E(S)$, then $x_1 x_2 \cdots x_{2k} y \in \langle y \rangle \subseteq A$, since $S/E(S)$ is an $\mathcal{L}_{2k}$-semigroup. □

By analogy with the Theorem 3. we have the following

THEOREM 4. *The following conditions on a semigroup S are equivalent:*

(i) *S is an $\mathfrak{L}_{2k+1}$-semigroup;*

(ii) $(\forall x_1, x_2, \dots, x_{2k+1}, y \in S)\ x_1 x_2 \cdots x_{2k+1} y \in \{y^2, y^3, \dots, y^{2k+3}\}$;

(iii) *S is a retractive extension of a right zero band by an $\mathfrak{L}_{2k+1}$-nil-semigroup.* □

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