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SOME CONGRUENCES ON A π-REGULAR SEMIGROUP

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Abstract. *The group congruences on an eventually regular (π-regular) semigroups is described in* [5]. *In this paper, by the method from* [5], *we give some new descriptions for group congruences. Also, we consider normal congruences on a semigroup which is generated by all the idempotents from a π-regular semigroup. If S is a regular semigroup and idempotents of S form a left zero band, then we define the normal congruence pair and by it we describe some congruences. For the related results see* [7].

1. Introduction

A semigroup S is *π-regular* (eventually regular) if for every $a \in S$ there exists a positive integer m such that $a^m \in a^m S a^m$. We shall denote by $Reg(S)$ the set of all regular elements of S and by $E(S)$ the set of all idempotents of S. If x is a regular element of a semigroup S, $V(x)$ will denotes the set of inverses of x. A mapping $r : S \to Reg(S)$ is defined with $r(a) = a^n$, where n is the least positive integer for which $a^n \in Reg(S)$, [8].

If $\mathcal{A}$ is a class of semigroups, then a congruence ρ on a semigroup S is an *$\mathcal{A}$-congruence* if $S/\rho \in \mathcal{A}$.

For undefined notions and notations we refer to [3] and [6].

2. Group congruences

In this section S will be arbitrary π-regular semigroup.

A subset H of S is defined to be *full* if $E(S) \subseteq H$. For any subset H of S the *closure* $H\omega$ of H is the set $\{x \in S \mid hx \in H \text{ for some } h \in H\}$; H is said to be *closed* if $H\omega = H$.

A subset H of S is called *self-conjugate* if $aHa^{n-1}(a^n)' \subseteq H$ and $a^{n-1}(a^n)'Ha \subseteq H$ for all $a \in S$ and $(a^n)' \in V(a^n)$, $a^n = r(a)$, [5].

Lemma 2.1. [5] *If H is a full self-conjugate subsemigroup of an eventually regular semigroup S, then $H\omega = H$ if and only if for all $h \in H$ and $x \in S$,*

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$xh \in H$ implies $x \in H$. □

LEMMA 2.2. *If H is a full, self-conjugate closed subsemigroup of S, then for each $a \in S$, $(a^n)' \in V(a^n)$, $a^n = r(a)$ holds*

$$aHa^{n-1}(a^n)' \subseteq H \iff aH(a^n)'a^{n-1} \subseteq H\ ,$$
$$a^{n-1}(a^n)'Ha \subseteq H \iff (a^n)'a^{n-1}Ha \subseteq H\ .$$

PROOF. Let $aHa^{n-1}(a^n)' \subseteq H$. Since $(a^n)'a^n \in E(S) \subseteq H$, then $H(a^n)'a^{n-1}a \subseteq H$ and $a(H(a^n)'a^{n-1}a)a^{n-1}(a^n)' \subseteq H$. Since $a^n(a^n)' \in E(S)$, then $aH(a^n)'a^{n-1} \subseteq H$.

Conversely, let $aH(a^n)'a^{n-1} \subseteq H$. Since $a^{n-1}(a^n)'a \in E(S)$, we have $Ha^{n-1}(a^n)'a \subseteq H$ and $a(Ha^{n-1}(a^n)'a)(a^n)'a^{n-1} \subseteq H$. Since $a(a^n)'a^{n-1} \in E(S)$, then $aHa^{n-1}(a^n)' \subseteq H$.

The second part of this lemma can be proved similarly. □

We denote with

$$\mathcal{B} = \{H \subseteq S \mid H \text{ is a full, self-conjugate closed subsemigroup of } S\}\ .$$

Then $\mathcal{B} \neq \emptyset$ since the kernel of any group congruence on S is an element of $\mathcal{B}$.

THEOREM 2.1. [5] *If $H \in \mathcal{B}$ then*

$$\beta_H = \{(a,b) \in S \times S \mid ab^{n-1}(b^n)' \in H \ \text{ where } (b^n)' \in V(b^n),\ b^n = r(b)\}$$

is a group congruence on S. □

If $H \in \mathcal{B}$, then $\ker\beta_H = H$. If γ is a group congruence on S, then $\beta_{\ker\gamma} = \gamma$. The mapping $H \to \beta_H$ is an inclusion preserving one-to-one correspondence of the set $\mathcal{B}$ and the set of all group congruences on S, [5].

THEOREM 2.2. *Let $H \in \mathcal{B}$, then the following statements are equivalent:*

(1) $a(b^n)'b^{n-1} \in H$ *where* $(b^n)' \in V(b^n)$, $b^n = r(b)$;
(2) $b(a^m)'a^{m-1} \in H$ *where* $(a^m)' \in V(a^m)$, $a^m = r(a)$;
(3) $(a^m)'a^{m-1}b \in H$ *where* $(a^m)' \in V(a^m)$, $a^m = r(a)$;
(4) $(b^n)'b^{n-1}a \in H$ *where* $(b^n)' \in V(b^n)$, $b^n = r(b)$;
(5) $a\ \beta_H\ b$;
(6) $ax(b^n)'b^{n-1} \in H$ *for some* $x \in H$ *and* $(b^n)' \in V(b^n)$, $b^n = r(b)$;
(7) $bx(a^m)'a^{m-1} \in H$ *for some* $x \in H$ *and* $(a^m)' \in V(a^m)$, $a^m = r(a)$;
(8) $(a^m)'a^{m-1}xb \in H$ *for some* $x \in H$ *and* $(a^m)' \in V(a^m)$, $a^m = r(a)$;
(9) $(b^n)'b^{n-1}xa \in H$ *for some* $x \in H$ *and* $(b^n)' \in V(b^n)$, $b^n = r(b)$.

PROOF. (1) $\Longrightarrow$ (2). Let $a(b^n)'b^{n-1} \in H$. Then by Lemma 2.2. we have $(b^n)'b^{n-1}a(b^n)'b^{n-1}b \in H$ and $a(b^n)'b^{n-1}a(b^n)'b^{n-1}b(a^m)'a^{m-1} \in H$. Since H is closed we have $b(a^m)'a^{m-1} \in H$.

(2) $\Longrightarrow$ (3). Let $b(a^m)'a^{m-1} \in H$, then $b^{n-1}(b^n)'b(a^m)'a^{m-1}b \in H$ and since $b^{n-1}(b^n)'b \in E(S) \subseteq H$ we have $(a^m)'a^{m-1}b \in H$.

(3) $\Longrightarrow$ (4). If $(a^m)'a^{m-1}b \in H$, then $b(a^m)'a^{m-1}b(b^n)'b^{n-1} \in H$ and $(a^m)'a^{m-1}b(a^m)'a^{m-1}b(b^n)'b^{n-1}a \in H$, whence $(b^n)'b^{n-1}a \in H$.

(4) $\Longrightarrow$ (1). If $(b^n)'b^{n-1}a \in H$, then $a(b^n)'b^{n-1}a(a^m)'a^{m-1} \in H$ and since $a(a^m)'a^{m-1} \in E(S)$ and H is closed we have $a(b^n)'b^{n-1} \in H$.

(1) $\Longrightarrow$ (5). If $a(b^n)'b^{n-1} \in H$, then $b(a^m)'a^{m-1}a(b^n)'b^{n-1}ab^{n-1}(b^n)' \in H$. Now, by (2) $b(a^m)'a^{m-1} \in H$ and since H is closed we have $ab^{n-1}(b^n)' \in H$. Hence, $a\ \beta_h\ b$.

(5) $\Longrightarrow$ (1). Let $a\ \beta_H\ b$, then $a^{m-1}(a^m)'ab^{n-1}(b^n)'a \in H$ and $ba^{m-1}(a^m)'a$ $b^{n-1}(b^n)'a(b^n)'b^{n-1} \in H$. Now $ba^{m-1}(a^m)' \in H$ since β_H is symmetric and so $a(b^n)'b^{n-1} \in H$.

(6) $\Longrightarrow$ (7). Let $ax(b^n)'b^{n-1} \in H$. Since $x(b^n)'b^{n-1}bx \in H$ and since $ax(b^n)'b^{n-1}bx(a^m)'a^{m-1} \in H$ we have $bx(a^m)'a^{m-1} \in H$.

(7) $\Longrightarrow$ (8). If $bx(a^m)'a^{m-1} \in H$ then $bx(a^m)'a^{m-1}x \in H$ and $b^{n-1}(b^n)'bx$ $(a^m)'a^{m-1}xb \in H$, so $(a^m)'a^{m-1}xb \in H$.

(8) $\Longrightarrow$ (9). Let $(a^m)'a^{m-1}xb \in H$. Since $xa(a^m)'a^{m-1}x \in H$ and $(b^n)'b^{n-1}xa(a^m)'a^{m-1}xb \in H$ we have $(b^n)'b^{n-1}xa \in H$.

(9) $\Longrightarrow$ (6). Let $(b^n)'b^{n-1}xa \in H$. Since $(b^n)'b^{n-1}xax \in H$ and $b(b^n)'b^{n-1}xax(b^n)'b^{n-1} \in H$ we have $ax(b^n)'b^{n-1} \in H$.

(1) $\Longrightarrow$ (9). If $a(b^n)'b^{n-1} \in H$, then $a(b^n)'b^{n-1}x \in H$ for $x \in H$ and $a^{m-1}(a^m)'a(b^n)'b^{n-1}xa \in H$, so $(b^n)'b^{n-1}xa \in H$.

(9) $\Longrightarrow$ (1). If $(b^n)'b^{n-1}xa \in H$, then $a(b^n)'b^{n-1}xa(a^m)'a^{m-1} \in H$. Since $xa(a^m)'a^{m-1} \in H$ we have $a(b^n)'b^{n-1} \in H$, which completes the proof. □

The statements (1)-(9) from Theorem 2.2. are equivalent with statements (2)-(11) from Theorem 3. [5].

3. Normal congruences on a semigroup $\langle E(S)\rangle$

If S is a π-regular semigroup, then by $\langle E(S)\rangle$ we denote the subsemigroup of S generated by $E(S)$.

DEFINITION 3.1. *Let S be a π-regular semigroup. An equivalence (congruence) ξ on the semigroup $\langle E(S)\rangle$ is a normal equivalence(congruence) if for every $x, y \in \langle E(S)\rangle$, $a \in S$ and $(a^n)' \in V(a^n)$, $a^n = r(a)$ holds*

$$x\ \xi\ y \implies ax(a^n)'a^{n-1}\ \xi\ ay(a^n)'a^{n-1}$$

whenever $ax(a^n)'a^{n-1}, ay(a^n)'a^{n-1} \in \langle E(S)\rangle$.

If S is a π-regular semigroup and the set $E(S)$ is a subsemigroup of S, then $ae(a^n)'a^{n-1} \in E(S) = \langle E(S)\rangle$ for each $e \in E(S)$, $a \in S$ and $(a^n)' \in V(a^n)$, $a^n = r(a)$.

THEOREM 3.1. *Let S be a π-regular semigroup, ξ be a normal congruence on $\langle E(S)\rangle$ and let α be the relation on $\langle E(S)\rangle$ defined with*

$$x\ \alpha\ y \iff ax(a^n)'a^{n-1}\ \xi\ ay(a^n)'a^{n-1} \tag{3.1}$$

for each $a \in S$, $(a^n)' \in V(a^n)$, $a^n = r(a)$, whenever $ax(a^n)'a^{n-1}, ay(a^n)'a^{n-1} \in \langle E(S)\rangle$. Then the relation α is a normal equivalence on $\langle E(S)\rangle$ and $\xi \subseteq \alpha$.

PROOF. Clearly, α is an equivalence. Let $x, y \in \langle E(S)\rangle$ and $x\ \alpha\ y$, then since ξ is a normal congruence we have

$$b(ax(a^n)'a^{n-1})(b^m)'b^{m-1}\ \xi\ b(ay(a^n)'a^{n-1})(b^m)'b^{m-1}\ ,$$

for every $b \in S$, $(b^m)' \in V(b^m)$, $b^m = r(b)$, whenever

$$b(ax(a^n)'a^{n-1})(b^m)'b^{m-1},\ b(ay(a^n)'a^{n-1})(b^m)'b^{m-1} \in \langle E(S)\rangle \ .$$

Hence, $ax(a^n)'a^{n-1} \ \alpha \ ay(a^n)'a^{n-1}$ and α is a normal equivalence on $\langle E(S)\rangle$. From $x, y \in \langle E(S)\rangle$ and $x \ \xi \ y$ it follows that $ax(a^n)'a^{n-1} \ \xi \ ay(a^n)'a^{n-1}$ and so $x \ \alpha \ y$, whence $\xi \subseteq \alpha$. □

A band B is a *rectangular band* if for every $e, f, g \in B$ holds $efg = eg$. A band B is a *right regular band* if for every $e, f \in B$ holds $ef = fef$.

Corollary 3.1. *If S is a π-regular semigroup and $E(S)$ is a rectangular band (right regular band), then α is a normal congruence on $E(S)$.*

Proof. Let $E(S)$ be a rectangular band, $e, f, g, h \in E(S)$ and

$$e\alpha = f\alpha \iff (ae(a^n)'a^{n-1})\xi = (af(a^n)'a^{n-1})\xi \ ,$$
$$g\alpha = h\alpha \iff (ag(a^n)'a^{n-1})\xi = (ah(a^n)'a^{n-1})\xi \ ,$$

where $a \in S$, $(a^n)' \in V(a^n)$, $a^n = r(a)$. Since ξ is a congruence, we have

(3.2) $$(ae(a^n)'a^{n-1}ag(a^n)'a^{n-1})\xi = (af(a^n)'a^{n-1}ah(a^n)'a^{n-1})\xi$$

whence it follows that

(3.3) $$(aeg(a^n)'a^{n-1})\xi = (afh((a^n)'a^{n-1})\xi \ .$$

Hence, $(eg)\alpha = (fh)\alpha$, so α is a congruence on $E(S)$.

If $E(S)$ is a right regular band, then from (3.2) we have

$$(ae(a^n)'a^n g(a^n)'a^n(a^n)'a^{n-1})\xi = (af(a^n)'a^n h(a^n)'a^n(a^n)'a^{n-1})\xi \ ,$$

whence it follows (3.3). □

Theorem 3.2. *Let S be a π-regular semigroup and ξ is a normal congruence on $\langle E(S)\rangle$. Then the relation τ on $\langle E(S)\rangle$ defined by:*

$$x \ \tau \ y \iff (\forall z \in \langle E(S)\rangle) \ xz \ \xi \ yz$$

is a normal congruence on $\langle E(S)\rangle$ and $\xi \subseteq \tau \subseteq \alpha$, where α is defined with (3.1).

Proof. Let $x, y \in \langle E(S)\rangle$. Then

$$x \ \tau \ y \implies (\forall z, t \in \langle E(S)\rangle) \ xtz \ \xi \ ytz,\ txz \ \xi \ tyz \ ,$$

and $xt \ \tau \ yt$, $tx \ \tau \ ty$. Hence, τ is a congruence.

Let $x \ \xi \ y$. Then $xz \ \xi \ yz$ for every $z \in \langle E(S)\rangle$ and so $x \ \tau \ y$. Hence, $\xi \subseteq \tau$. Also,

(3.4) $$x \ \tau \ y \implies x(a^n)'a^n \ \xi \ y(a^n)'a^n$$
$$\implies a(x(a^n)'a^n)(a^n)'a^{n-1} \ \xi \ a(y(a^n)'a^n)(a^n)'a^{n-1}$$

(3.5) $$\implies ax((a^n)'a^{n-1} \ \xi \ ay(a^n)'a^{n-1}$$
$$\iff x \ \alpha \ y \ ,$$

and so $\tau \subseteq \alpha$. From (3.4), (3.5) and $\xi \subseteq \tau$ we have

$$x \ \tau \ y \implies ax(a^n)'a^{n-1} \ \tau \ ay(a^n)'a^{n-1} \ ,$$

and consequently τ is a normal congruence. □

A band B is *left regular* if for every $e, f \in B$ holds $ef = efe$. A π-regular semigroup S is *π-R-unipotent* if the set $E(S)$ is a left regular band. A band B is *normal* if $efgh = egfh$ for every $e, f, g, h \in B$.

COROLLARY 3.2. *If ξ is a normal and a normal band congruence on $\langle E(S)\rangle$, where S is a π-regular semigroup, then τ is a π-R-unipotent congruence on $\langle E(S)\rangle$.*

PROOF. Let ξ be a normal and a normal band congruence on $\langle E(S)\rangle$ and $A, B \in E(\langle E(S)\rangle /\tau)$. Then there exist $e, f \in E(S)$ [2,4], so that $A = e\tau$, $B = f\tau$. Now

$$\begin{aligned} AB = AB &\iff (ef)\tau = (ef)\tau \iff (\forall z \in \langle E(S)\rangle)(efz)\xi = (eefz)\xi \\ &\iff (\forall z \in \langle E(S)\rangle)\ (efz)\xi = (efez)\xi \\ &\iff (ef)\tau = (efe)\tau\ . \end{aligned}$$

Hence, $AB = ABA$ and so $\langle E(S)\rangle /\tau$ is a π-R-unipotent semigroup. □

For some similar results in a regular case we refer to [1].

4. A normal congruence pair

In this section S will be a regular semigroup and $E(S)$ will be a left zero band.

By Definition 3.1. the congruence ξ on $E(S)$ is normal if

$$e\ \xi\ f \iff aea'\ \xi\ afa'$$

for every $e, f \in E(S)$, $a \in S$ and $a' \in V(a)$.

DEFINITION 4.1. *If K is a full, self-conjugate and closed subsemigroup on S and ξ is a normal congruence on $E(S)$, then (ξ, K) is a normal congruence pair for S.*

THEOREM 4.1. *If (ξ, K) is a normal congruence pair for S, then the relation $\mathcal{K}_{(\xi,K)}$ defined on S by*

$$a\ \mathcal{K}_{(\xi,K)}\ b \iff (\exists a' \in V(a))(\exists b' \in V(b))\ aa'\ \xi\ bb',\ ab' \in K$$

is a congruence on S with a trace ξ and a kernel K.

PROOF. If we denote $a\ \delta\ b$ iff $aa'\ \xi\ bb'$, and $a\ \beta_K\ b$ iff $ab' \in K$, then

$$a\ \mathcal{K}_{(\xi,K)}\ b \iff a\ \delta\ b,\ a\ \beta_K\ b\ .$$

By Theorem 2.1, the relation β_K is a congruence on S. Clearly, the relation δ is an equivalence relation. Let $a\ \delta b$, $c \in S$. Since ξ is a normal congruence we then have

$$ca(ca)' = c(aa')c'\ \xi\ c(bb')c' = cb(cb)'\ ,$$

so $ca\ \delta\ cb$. Similarly,

$$\begin{aligned} ac(ac)' &= acc'a' = aa'acc'a' = aa'aa' = aa' \\ &\xi bb' = bb'bb' = bb'bcc'b' = bcc'b' = bc(bc)'\ , \end{aligned}$$

so $ac\ \delta\ bc$. Hence, δ is a congruence and also $\mathcal{K}_{(\xi,K)}$ is a congruence.

Let $e, f \in E(S)$. Then

$$e\ \mathcal{K}_{(\xi,K)}\ f \iff (\exists e' \in V(e))(\exists f' \in V(f))\ ee'\ \xi\ ff',\ ef' \in K\ .$$

Since $E(S)$ is a left zero band and $E(S) \subseteq K$, we have $e\ \mathcal{K}_{(\xi,K)}\ f$ if and only if $e\ \xi\ f$ and so $\mathrm{tr}\mathcal{K}_{(\xi,K)} = \xi$.

Let $a \in \ker\mathcal{K}_{(\xi,K)}$. Then there exists $e \in E(S)$ such that $ae' \in K$ for every $e' \in V(e) = E(S)$. Now

$$ae' = aa'ae' = aa'a = a \in K \ ,$$

and so $\ker\mathcal{K}_{(\xi,K)} \subseteq K$. If $a \in K$, then from $a = aa'a$ and $aa' = aa'aa'$ we have that $a\ \mathcal{K}_{(\xi,K)}\ aa'$ and so $a \in \ker\mathcal{K}_{(\xi,K)}$. □

THEOREM 4.2. *If ρ is a congruence on S, then $(tr\rho, \ker\rho)$ is a normal congruence pair for S.*

PROOF. Let ρ be a congruence on S. A simple verification shows that $\ker\rho$ is a full, self-conjugate subsemigroup of S. Let $h, xh \in \ker\rho$. Then $h\rho = e\rho$, $(xh)\rho = f\rho$ for some $e, f \in E(S)$. Since $E(S)$ is a left zero band we have

$$f\rho = (xh)\rho = x\rho h\rho = (xx'x)\rho e\rho = x\rho(xx'e)\rho = x\rho(xx')\rho = x\rho \ ,$$

where $x' \in V(x)$, and so $x \in \ker\rho$. Hence, $\ker\rho$ is a closed subsemigroup. Since $\mathrm{tr}\rho = \rho_{|E(S)}$ is a normal congruence on $E(S)$, we have that $(\mathrm{tr}\rho, \ker\rho)$ is a normal congruence pair for S. □

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