Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 6 (1992), 169–174

FILOMAT–20, Niš, September 26–28, 1991

RELATIVELY COMPLETE FINITE EXTENSIONS OF BOOLEAN ALGEBRAS

Žikica Perović

Abstract. *In this paper we characterize relatively complete finite extensions of Boolean algebras by the independent sets of its generators. Particularly, we construct a minimal set of generators for a given finite rc-extension.*

0. Introduction

Let C be a subalgebra of B. We say that $q \in \mathrm{Ult}\, C$ splits in B if there are distinct $p, p' \in \mathrm{Ult}\, B$ which extend q i.e. $p \cap C = p' \cap C = q$. Let C and B be Boolean algebras. C is relatively complete (rc) subalgebra of B if for each $b \in B$ there is a greatist element $c \in C$ such that $c \leq b$. We denote that element by $\mathrm{pr}_C(b)$. We also denote by $\mathrm{indp}_C(b) = -(\mathrm{pr}(b) + \mathrm{pr}(-b))$. It is a clopen set in $\mathrm{Ult}\, C$ consisting of points that have at least one extension to an ultrafilter of B containg a, and at least one containg $-a$. B is a 2-extension of C if every ultrafilter in $\mathrm{Ult}\, C$ has at most two extensions to an ultrafilter on B. B is an rcs extension of C if it is a relatively complete simple extension, i.e. extension by one element. B is a finite extension of C if there exist $u_1, \dots, u_n \in B$, such that $B = C(u_1, \dots, u_n)$.

In the following proposition we list some known facts. Proofs could be found in (2).

Proposition 0 *Let B be an rc-extension of C.*

i) *If B is an rcs-extension of C then it is a 2-extension .*

ii) *Let $U = \{q \in \mathrm{Ult}\, C \mid q$ splits in $B\}$. Then $U = \cup\{s(j) \mid j \in J\}$ where $s : C \to \mathrm{ClopUlt}\, C$ is the Stone isomorphism. In particular U is open in $\mathrm{Ult}\, C$.*

iii) *$J = \{\mathrm{indp}_C(x) \mid x \in B\}$ is an ideal in C, in fact the ideal dual to $U \in \mathrm{Ult}\, C$.*

iv) *Let α, β, γ be pairwise disjoint elements of C such that $\alpha + \beta + \gamma = 1$ and $\alpha \in J$. Assume $x \in A$ and $\mathrm{indp}(x) \leq \alpha$. Then there is some $z \in A$ such that $\mathrm{indp}(z) = \alpha$, $\mathrm{pr}(z) = \beta$, $\mathrm{pr}(-z) = \gamma$ and $x \in C(z)$.*

v) *Let $\mathrm{indp}(a) = C$. Then $C(a) \cong C \oplus 4$.*

vi) *Let C be a Boolean algebra and $\alpha \in C$. There exists an rcs extension $B = C(a)$ of C such that $\mathrm{indp}(a) = \alpha$.*

vii) *If $b \in C(a)$ then $\mathrm{indp}(b) \leq \mathrm{indp}(a)$ and the equality holds iff $C(b) = C(a)$.*

Supported by Grant 0401A *of Science Fund of Serbia through Mat. Inst. SANU*
1991 *Mathematics subject classification. Primary:* 06E05

viii) *If* $C(a)$ *and* $C(b)$ *are two rcs extensions of* C *such that* $\mathrm{indp}(b) = \mathrm{indp}(a)$ *then there is an isomorphism* $f : C(a) \to C(b)$ *such that* $f|C = id_C$ *and* $f(a) = b$.

ix) *If* B *is an rcs-extension of* C *then* U_C^B *is clopen.*

x) *Canonical mapping* $f : \mathrm{Ult}\, B \to \mathrm{Ult}\, C$ *is open .*

1. Dual atom rc-extensions

DEFINITION. *B is a dual atom extension of C if C is a dual atom subalgebra of B i.e. there exists $b \in B$ such that C is the maximal subalgebra of B which does not contain b .*

In the following proposition we give another proof of a well known fact that dual atom extensions are simple for the special case of rc-extensions. It is much shorter than the proof in general case.

PROPOSITION 1 *Dual atom rc-extension is an rcs-extension.*

PROOF. Let $C < B$ and let $b \in B$ so that C is the maximal subalgebra of B not containing b. $\mathrm{indp}(b) \neq 0$. Let $c \in B \setminus C$. Since $b \in C(c)$, $\mathrm{indp}(b) \leq \mathrm{indp}(c)$. $\mathrm{indp}(c) \leq \mathrm{indp}(b)$ since otherwise for $d \in B$ such that $\mathrm{indp}(d) = \mathrm{indp}(c) - \mathrm{indp}(b)$, $C(d)$ would be a proper subalgebra of B properly containing C (Proposition o.vii)) and not containing b . Hence $C(b) = C(c)$ (Proposition 0.vii) , so $c \in C(b)$. Since c was arbitrary, $C(b) = B$.

PROPOSITION 2 *Let $C <_{rc} B$ and let B be a dual atom extension of C. B is an extension by an element whose independent part is an atom of C. Any such rc extension is a dual atom extension.*

PROOF. We use notation from the preceeding proposition. We prove that $\mathrm{indp}(b)$ is an atom of C. Let $\alpha \in C$ and $0 < \alpha < \mathrm{indp}(b)$. Choose $a \in B$ so that $\mathrm{indp}(a) = \alpha$ (Proposition 0.iv). Then $C(a)$ is a proper subalgebra of B , and proper extension of C , but not containing b. Contradiction.

For the other direction, if $\mathrm{indp}(b)$ is an atom of C, then for every element $a \in C(b)$, $\mathrm{indp}(a) \leq \mathrm{indp}(b)$, hence it is 0 or $\mathrm{indp}(b)$. Hence $C(a) = C$ or $C(a) = C(b)$. So we have that $b \notin C$, and for every $a \in B \setminus C$, $b \in C(a)$.

CORROLARY *C has an rc dual atom extension iff it has atoms.*

2. Finite rc-extensions

Let $B = C(u_1, \ldots, u_n)$. Set of generators $F = \{u_1, \ldots, u_n\}$ is reduced if they are partition of one and for every $u \neq v \in F, u \in \langle C \cup F \setminus \{u, v\}\rangle$. $J_i^u = \{a \in C \mid au_i = 0\}$, $i \leq n$. These principal ideals make an extender meaning that their intersection contains just 0, and if $a \in C$ belongs to one of them then $-a$ does not belong to any of them.

PROPOSITION 1 *Let $B = C(u_1, \ldots, u_n)$ be an rc-extension, $\langle u_1, \ldots, u_n\rangle$ reduced, $J_i^u = (\alpha_i)$.*

i) $\mathrm{pr}(u_i) = \wedge\{\alpha_j \mid j \neq i\}$

ii) $\mathrm{indp}(u_i) = 1 - \alpha_i - \wedge\{\alpha_j \mid j \neq i\}$.

PROPOSITION 2 *Let $B = C(u_1, \dots, u_n)$ be an rc-extension, $\{u_1, \dots, u_n\}$ disjoint, nonreduced. There exists a reduced set of generators for B over C, of smaller cardinality.*

PROOF. Suppose $u_1 \in \langle C \cup \{u_3, \dots, u_n\}\rangle$. Then $u_1 = a_3u_3 + \dots + a_nu_n + a(u_1+u_2)$, and similarly u_2 . Hence $\{u_1+u_2, \dots, u_n\}$ is still disjoint set of generators for B over C having $n-1$ elements. If it is still not reduced we repeat construction untill it finally stops, when we get a reduced set of generators.

The following proposition is Theorem 3.1 from (1):

PROPOSITION 3.

i) *Let $\langle u_i \mid i \le m\rangle$ be reduced in $C(u_1, \dots, u_n)$. Then $\langle J_i^u \mid i \le n\rangle$ make an extender.*

ii) *Conversely, let $\langle K_i \mid i \le n\rangle$ be an extender. Then there is an extension B of C and a reduced sistem $\langle u_i \mid i \le n\rangle$ in B such that $B = C(u_1, \dots, u_n)$ and $J_i^u = K_i, i \le n$. If $A = C(v_1, \dots, v_n)$ with $\langle v_i \mid i \le n\rangle$ a partition of unity and $J_i^v = K_i, i \le n$, then there is an isomorphism g of B onto A such that $g|A = id$ and $g(u_i) = v_i$, $i \le n$.*

PROPOSITION 4.

(i) *Let $B = C(u_1, \dots, u_n)$ and $A = C(v_1, \dots, v_n), \langle u_1, \dots, u_n\rangle$ and $\langle v_1, \dots v_n\rangle$ reduced and* $\operatorname{indp}(u_i) = \operatorname{indp}(v_i)$, $i \le n$. *There exists an isomorphism $\varphi : B \to A$ such that $\varphi|A = id$. More precisely,*

(ii) *There exist $\langle u_1', \dots, u_n'\rangle$ reduced , and $\langle v_1', \dots, v_n'\rangle$ reduced, such that* $\operatorname{indp}(u_i') = \operatorname{indp}(u_i)$, $\operatorname{indp}(v_i') = \operatorname{indp}(v_i)$, $i \le n$, $B = C(u_1', \dots, u_n')$, $A = C(v_1', \dots, v_n')$, *and isomorphism $\varphi : B \to A$ such that $\varphi|A = id$ and $\varphi(u_i') = \varphi(v_i')$.*

PROOF. Let $u_i' = u_i - \operatorname{pr}(u_i)$, $i \ne 1$, and $u_1' = u_1 + \sum\{\operatorname{pr}(u_i) \mid i \ge 2\}$. v_i' are constructed the same way from v_i. It is obvious that $u_1', \dots, u_n'$ and $v_1', \dots, v_n'$ are reduced and $\operatorname{indp}(u_i') = \operatorname{indp}(u_i), \operatorname{indp}(v_i') = \operatorname{indp}(v_i)$, $i \le n$, $B = C(u_1', \dots, u_n')$, $A = C(v_1', \dots, v_n')$. Let $J_i^u = (\alpha_i)$ and $J_i^v = (\beta_i)$. Then, since $\operatorname{pr}(u_i) = \operatorname{pr}(v_i) = 0$, $i \ge 2$, we have $\alpha_i = 1 - \operatorname{indp}(u_i')$, $\beta_i = 1 - \operatorname{indp}(v_i')$, $i \ge 2$, so $\alpha_i = \beta_i$ for $i \ge 2$. Since $\operatorname{pr}(u_1') = \wedge\{\alpha_i \mid i \ge 2\}$, and $\operatorname{pr}(v_1') = \wedge\{\beta_i \mid i \ge 2\}$, we have $\operatorname{pr}(u_1') = \operatorname{pr}(v_1')$. Henceforth $\alpha_1 = 1 - \operatorname{pr}(u_1') - \operatorname{indp}(u_1') = \beta_1$. Therefore by Proposition 3 there exists an isomorphism from the statement.

PROPOSITION 5. *Let $B = C(u_1, \dots, u_n), u_1, \dots, u_n$ disjoint. If* $\operatorname{indp}(u_i)\operatorname{indp}(u_j)$ $\ne 0$ *for $1 \le i < j \le n$, $\langle u_1, \dots, u_n\rangle$ is reduced.*

PROOF. It is enough to show that $\langle J_i^u \mid 1 \le i \le n\rangle$ is an extender. But if for $a \in C, a \in J_i^u$ and $-a \in J_j^u$, then $\operatorname{indp}(u_i) \le -a$ and $\operatorname{indp}(u_j) \le a$, contrary to assumption that they are not disjoint.

DEFINITION. *Let $B = C(u_1, \dots, u_n)$, $\langle u_1, \dots, u_n\rangle$ reduced. For $p \in \operatorname{Ult} C$, $h(p)$ is the number of extensions of p in* Ult B.

PROPOSITION 6 *Let C and B be as in definition, and let for $p \in \operatorname{Ult} C$ which splits in B, $M_p = \{i \mid i \le n, p \in \operatorname{indp}(u_i)\}$. $h(p) = |M_p|$.*

PROOF. Let $\langle p\rangle^{fi}$ denote a filter of B generated by p, $B_p = B/\langle p\rangle^{fi}$, and $\varphi : B \to B_p$ the canonical mapping. Since $\{u_1, \dots, u_n\}$ make a partition in B,

$\{u_1/\langle p\rangle^{fi}, \ldots, u_n/\langle p\rangle^{fi}\}$ make a partition in B_p . Nonzero ones are atoms of B_p since for any $b \in B$, if $b = \alpha_1 u_1 + \cdots + \alpha_n u_n$ then $\varphi(b) \leq \varphi(u_i)$ implies $\varphi(b) = \varphi(\alpha_i)\varphi(u_i)$ i.e. $\varphi(b)$ is $\varphi(u_i)$ or 0 depending on wether $\alpha_i \in p$. On the other hand , since extensions of p to B are in one-one correspondence with $\mathrm{Ult}\, B_p$,

$$h(p) = |\,\mathrm{Ult}\, B_p| = |\,\mathrm{At}(B_p)| = |\{i \mid i \leq n,\ \varphi(u_i) \neq 0\}| = |M_p|.$$

Corrolary *Let $B = C(u_1, \ldots, u_n), \langle u_1, \ldots, u_n\rangle$ reduced. $h(p) \leq n$.*

Definition. *Let $B = C(u_1, \ldots, u_n)$. $\mathcal{F}_k^B = \{p \in \mathrm{Ult}\, C \mid h(p) = k\}$, $k \leq n$.*

Proposition 7 *$\mathcal{F}_k^B$ is clopen in $\mathrm{Ult}\, C$ i.e. $\mathcal{F}_k^B \in C$, and $\vee\{\mathcal{F}_k^B \mid k \leq n\} = 1$.*

Proof. Let $p \in \mathcal{F}_k^B$. Let $F = \{i \leq n \mid p \in \mathrm{indp}(u_i)\}$. $|F| = k$. Then $p \in \bigcap\{\mathrm{indp}(u_i) \mid i \in F\} \cap \bigcap\{\mathrm{indp}(u_i)' \mid i \notin F\} \subseteq \mathcal{F}_k^B$. Hence $\mathcal{F}_k^B$'s are open. Since their union is $\mathrm{Ult}\, C$, they are all clopen.

Proposition 8 *Let $C <_{rc} A <_{rc} B$ be finite extensions.*
(i) *$h_A(p) \leq h_B(p)$, for every $p \in \mathrm{Ult}\, C$*
(ii) *$A = B$ iff equality holds for every $p \in \mathrm{Ult}\, C$.*

Proof.

(i) Let $\varphi_A : \mathrm{Ult}\, A \to \mathrm{Ult}\, C$ and $\varphi_B : \mathrm{Ult}\, B \to \mathrm{Ult}\, C$ be canonical mappings. Let $h_A(p) = k$. Then p extends to k ultrafilters in $\mathrm{Ult}\, B$. But each of them extends to at least one ultrafilter from $\mathrm{Ult}\, B$, which are extensions of p also, hence $h_B(p) \geq k$.

(ii) Suppose $A \neq B$. Then there exists an ultrafilter $r \in \mathrm{Ult}\, A$ which extends to two different ultrafilters $q_0, q_1 \in \mathrm{Ult}\, B$. Let $p = r \cap C$. $p \in \mathrm{Ult}\, C$. Suppose $h_A(p) = k$. Then there exist k ultrafilters $r = r_1, \ldots, r_k \in \mathrm{Ult}\, A$ which extend p. Each of ultrafilters $r_2, \ldots, r_k$ has an extension to $\mathrm{Ult}\, B$. Let it be $q_2, \ldots, q_k$. Ultrafilters $q_0, \ldots, q_k$ are different extensions of p to $\mathrm{Ult}\, B$, hence $h_B(p) \geq k+1$. Contradiction. Hence $A = B$.

Theorem 1 *Let $\{a_k \in C \mid k \leq n\}$ be a disjoint family having union 1, $a_n \neq 0$. There exists an extension $B = C(u_1, \ldots, u_n), \langle u_1, \ldots, u_n\rangle$ reduced such that $\mathcal{F}_k^B = a_k$, $k \leq n$.*

Proof. Let $\alpha_k = \vee\{a_i \mid i < k\}$, $1 \leq k \leq n$, and $K_k = (\alpha_k)$, $1 \leq k \leq n$. It is obviously an extender. By Proposition 3, there exists an extension $B = C(u_1, \ldots, u_n), \langle u_1, \ldots, u_n\rangle$ reduced, such that $J_k^u = K_k$, $k \leq n$. By Proposition 1, $\mathrm{pr}(u_1) = \alpha_2$, hence $\mathrm{indp}(u_1) = 1 - \alpha_2$. For $k \geq 2$ we have $\mathrm{pr}(u_k) = 0$, and $\mathrm{indp}(u_k) = 1 - \alpha_k$. Hence we have for $k \geq 2$, $p \in a_k$, $p \in \mathrm{indp}(u_i)$ iff $1 \leq i \leq k$. Henceforth $\mathcal{F}_k^B = a_k$.

Before we proceed to the second theorem we will prove a few lemmas.

Lemma 1 *Let $C <_{rc} B$, $\alpha \in C$, $u, v \in B$*
i) $\mathrm{indp}(\alpha u) = \alpha\, \mathrm{indp}(u)$
ii) $\mathrm{indp}(u + v) \leq \mathrm{indp}(u) + \mathrm{indp}(v)$.

PROOF.

(i) For $p \in \alpha$, $\alpha u / \langle p\rangle^{fi} = u / \langle p\rangle^{fi}$, and for $p \in \alpha'$ both sides become 0.

(ii) For $p \notin \operatorname{indp}(u) + \operatorname{indp}(v)$, if both $u / \langle p\rangle^{fi}$ and $v / \langle p\rangle^{fi}$ are 0 then $u + v / \langle p\rangle^{fi}$ is also 0, otherwise one of them is 1 and $u + v / \langle p\rangle^{fi}$ is also 1, hence $p \notin \operatorname{indp}(u + v)$.

LEMMA 2 *Let $C <_{rc} B$, $u, v \in B$, $\operatorname{pr}(u) = \operatorname{pr}(v) = 0$.*

(i) $(\operatorname{indp}(u) - \operatorname{indp}(v)) + (inpd(v) - \operatorname{indp}(u)) \leq \operatorname{indp}(u + v)$.

(ii) $\operatorname{indp}(u) \operatorname{indp}(v) = 0 \Rightarrow \operatorname{indp}(u + v) = \operatorname{indp}(u) + \operatorname{indp}(v)$.

PROOF.

i) If $p \in \operatorname{Ult} C$, and $p \in \operatorname{indp}(u) - \operatorname{indp}(v)$, then $v / \langle p\rangle^{fi} = 0$, hence $u + v / \langle p\rangle^{fi} = u / \langle p\rangle^{fi}$, hence $p \in \operatorname{indp}(u + v)$. The other case is considered similarly.

ii) Follows from (i) and Lema 1, (ii).

THEOREM 2 *Let B be a finite rc extension of C, such that* $\max\{h(p) \mid p \in \operatorname{Ult} C\} = l$. *There exists $\langle v_1, \ldots, v_l\rangle$ reduced, such that $B = C(v_1, \ldots, v_l)$. B cannot be generated by a smaller reduced set over C. If M is a generating set for B over C then $2^{|M|} \geq l$.*

PROOF. Let $B = C(u_1, \ldots, u_n)$, $\langle u_1, \ldots, u_n\rangle$ reduced. Wlog we can suppose that $\operatorname{pr}(u_1) = a_1$, and $\operatorname{pr}(u_i) = 0$ for $2 \leq i \leq n$. For $F \subseteq \{1, \ldots, n\}$, $a_{k,F} = a_k \wedge \bigwedge\{\operatorname{indp}(u_i) \mid i \in F\}$, and for $1 \leq i \leq k$, $F(i)$ will denote the i'th element of F ordered increasingly. $c(n, k)$ will denote the set of all subsets of $\{1, \ldots, n\}$ of power k.

$$v_1 = a_1 + \sum_{k=2}^{l} \sum_{F \in c(n,k)} a_{k,F} u_{F(1)}$$

$$v_i = \sum_{k=i}^{l} \sum_{F \in c(n,k)} a_{k,F} u_{F(i)}, \quad 2 \leq i \leq l.$$

$v_i's$ are disjoint since for different k's or F's , $a_{k,F}$' s are disjoint , and for the same k, F, $U_{F(i)}$'s are disjoint.

By Lemmas 1,2:

$$\operatorname{indp}(v_1) = \operatorname{indp}\left(\sum_{k=2}^{l} \sum_{F \in c(n,k)} a_{k,F} u_{F(1)}\right) = \sum_{k=2}^{l} \sum_{F \in c(n,k)} a_{k,F} \operatorname{indp}(u_{F(1)})$$

$$= \sum_{k=2}^{l} \sum_{F \in c(n,k)} a_{k,F} = \sum_{k=2}^{l} a_k.$$

Similarly $\operatorname{indp}(v_i) = \sum_{k=i}^{l} a_k$, $2 \leq i \leq l$.

Let $A = C(v_1, \ldots, v_l)$. Obviously $C < A < B$. Since for $2 \leq i \leq l$, $v_1, \ldots, v_i$ are disjoint and their independent parts include a_i, we have $h_A(p) \geq i$ for all

$p \in a_i$. Since $h_A(p) \leq h_B(p)$ for all $p \in \mathrm{Ult}\, C$ and $h_B(p) = i$ for $p \in a_i$, we have $h_A(p) = h_B(p)$, $p \in \mathrm{Ult}\, C$. Now, by Proposition 8 , we have $A = B$. $\langle v_1, \ldots, v_l \rangle$ are reduced since $\mathrm{indp}(v_i)\, \mathrm{indp}(v_j) \geq a_l$ (Proposition 5).
B cannot be generated by a smaller reduced set since reduced set should provide l atoms for $B/\langle p \rangle^{fi}$, for $p \in a_l$.
For the last statement, let $M = \{w_1, \ldots, w_m\}$ be a set of generators for B over C. Then for $A \subseteq \{1, \ldots, |M|\}$ let $u_A = w_1^{A(1)} ... w_m^{A(m)}$, where $A(i)$ is $'$ if $i \notin A$ and nothing if $i \in A$. It is obviously a disjoint set of generators for B over C, of cardinality at most 2^m. Therefore it could be reduced. Let $\{x_1, \ldots, x_s\}$ be its reduced form (Proposition 2). By the second part of this theorem we have $l \leq s \leq 2^m$.

CORROLARY *Finite rc2-extensions are simple.*

REFERENCES

[1] KOPPELBERG S., *Projective Boolean algebras*,in: D.Monk, ed., Handbook of Boolean algebras, 3, North Holand 1989, 741-775

[2] MONK D., *Automorphism groups*,in: D.Monk,ed., Handbook of Boolean algebras, 2, North Holand 1989, 517-546

[3] PEROVIĆ Ž ., *Relatively complete 2-extensions of Boolean algebras*, to appear.

Žikica Perović

Filozofski fakultet, Ćirila i Metodija 2,
18 000 Niš,
Yugoslavia