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A DECIDABLE POSITIVE RELEVANCE LOGIC

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ABSTRACT. *In this paper we define a positive relevance logic* $\mathbf{T'W_+}$ *and we show that it is decidable.*

$\mathbf{T'W_+}$ contains the positive fragment of the contractionless relevance logic $\mathbf{TW_+}$, and it is equivalent to the Gentzen-style system $\mathbf{G^uTW_+}$, which has been proved decidable in [3]. Moreover, $\mathbf{G^uTW_+}$ is presumably equivalent to $^u\mathbf{TW_+}$ investigated in [4].

Some systems close to $\mathbf{T'W_+}$ and $\mathbf{GT'W_+}$ were first formulated in [6]; however, the proof of Cut elimination theorem given there was incorrect, as noticed in [2].

The decidability proof in [3] was based on a Cut elimination theorem for $\mathbf{G^uTW_+}$. However, $\mathbf{G^uTW_+}$ was formulated there with a propositional constant **I** playing the role of a placeholder, which enabled the proof of the Cut elimination theorem for some other systems considered in [3] as well. Here we formulate a Gentzen-style system $\mathbf{GT'W_+}$, equivalent to $\mathbf{T'W_+}$, without **I**, and we give a direct Cut elimination proof. The proof given here does not work for $\mathbf{G^uRW}$ considered in [3]. Also, here we describe another decision procedure for $\mathbf{GT'W_+}$.

Proofs from hypotheses in $\mathbf{T'W_+}$

By $a, a_1, \dots, b, b_1, \dots$ we denote finite (possibly empty) sets of positive integers called *subscripts*. An ordered pair $\langle A, a\rangle$ is a called a *subscripted formula*, provided that A is a formula. In general we write A_a for $\langle A, a\rangle$. We omit the subscript $\emptyset$. Let $\max(a)$ denote the greatest element of a, if $a \neq \emptyset$; if $a = \emptyset$, then $\max(a) = 0$.

The axioms of $\mathbf{T'W_+}$ are those of $\mathbf{TW_+}$; they are given by the following schemes (compare [1], p. 340):

$$A \to A$$
$$A \to B \to .B \to C \to .A \to C$$
$$B \to C \to .A \to B \to .A \to C$$
$$A \& B \to A$$
$$A \& B \to B$$

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$$(A \to B)\&(A \to C) \to .A \to B\&C$$
$$A \to A \lor B$$
$$B \to A \lor B$$
$$(A \to C)\&(B \to C) \to .A \lor B \to C$$
$$A\&(B \lor C) \to A\&B \lor A\&C$$

The rules of **T'W$_+$** are modus ponens and adjunction (from A,B to infer $A\&B$).

By a *proof of* B_k from $\langle A_1, a_1\rangle, \ldots, \langle A_n, a_n\rangle$, $n \geq 0$, we mean a finite sequence of subscripted formulas $\langle B_1, b_1\rangle, \ldots \langle B_m, b_m\rangle$ such that B_b is $\langle B_m, b_m\rangle$ and for any $1 \leq k \leq m$ $\langle B_k, b_k\rangle$ is either a hypothesis $\langle A_j, a_j\rangle$, $1 \leq j \leq n$, or else an axiom or else a consequence of predecessors by adjunction or modus ponens. Furthermore,

(1) if $\langle B_k, b_k\rangle$ is a hypothesis, then $b_k \neq \emptyset$;

(2) if $\langle B_k, b_k\rangle$ is an axiom, then $b_k = \emptyset$;

(3) if $\langle B_k, b_k\rangle$ is a consequence of $\langle B_i, b_i\rangle$ and $\langle B_j, b_j\rangle$ by adjunction and $b_i = b_j$, then $b_i = b_k$;

(4) if $\langle B_k, b_k\rangle$ is a consequence of $\langle B_i, b_i\rangle$ and $\langle B_i \to B_k, b_j\rangle$ by modus ponens, then $b_k = b_i \cup b_j$.

The application of modus ponens is further restricted as follows: $b_i \cap b_j = \emptyset$, and either $\max(b_i) > \max(b_j)$ or $b_j = \emptyset$.

Let $X, Y, Z, \ldots$ range over (possibly empty) sequences of subscripted formulas. A subscripts a is called *used* in X iff there is member A_a of X. By $x, y, z, \ldots$ we denote the unions of all subscripts used in $X, Y, Z, \ldots$, respectively.

If there is a proof of A_a from hypotheses which are members of X, we write $X \vdash A_a$.

THEOREM 1. *The following rules are derived in* **T'W$_+$** *by using axioms, adjunction, modus ponens and the definition of a proof from hypotheses:*

P *From* $X, A_a, B_b, Y \vdash C_c$ *to infer* $X, B_b, A_a, Y \vdash C_c$;

W *From* $X \vdash B_b$ *to infer* $X, A_a \vdash B_b$, *provided that* $a \neq \emptyset$ *and* $a \subseteq b$;

C *From* $X, A_a, A_a \vdash B_b$ *to infer* $X, A_a \vdash B_b$;

TR *From* $X \vdash A_a$ *and* $Y, A_a \vdash B_b$ *to infer* $X, Y \vdash B_b$;

MP *From* $X \vdash A_a$ *and* $Y \vdash A \to B_b$ *to infer* $X, Y, \vdash B_{a \cup b}$, *provided that* $a \cap b = \emptyset$ *and either* $\max(a) > \max(b)$ *or* $b = \emptyset$;

CL *From* $X, A_a \vdash C_c$ *(from* $X, B_a \vdash C_c$*) to infer* $X, A\&B \vdash C_c$;

CR *From* $X \vdash A_a$ *and* $X \vdash B_a$ *to infer* $X \vdash A\&B_a$;

DL* *From* $X, A_a \vdash C_c$ *and* $X, B_a \vdash C_c$ *to infer* $X, A \lor B_a, \vdash C_c$, *provided that for any subscript* b *used in* X *either* $a = b$ *or* $a \cap b = \emptyset$ *and* $\max(a) > \max(b)$;

DR *From* $X \vdash A_a$ *(from* $X \vdash B_a$*) to infer* $X \vdash A \lor B_a$;

IL *From* $X \vdash A_a$ *and* $Y, B_{a \cup b} \vdash C_c$ *to infer* $X, Y, A \to B_b \vdash C_c$, *provided that* $a \cap b = \emptyset$ *and* $\max(a) > \max(b) > 0$;

IR *From* $X, A_a \vdash B_{a \cup b}$ *to infer* $X \vdash A \to B_b$, *provided that* $a \cap x = \emptyset$ *and* $\max(a) > \max(x)$.

PROOF. Left to the reader (for IR cf. [5]).

Let us consider the following rule:

DL *From $X, A_a \vdash C_c$ and $X, B_a \vdash C_c$ to infer $X, A \vee B_a \vdash C_c$.*

The system defined by the given axioms, adjunction and modus ponens is $\mathbf{TW}_+$, the positive fragment of contractionless relevance logic **TW**.

The system $\mathbf{T'W}_+$ is obtained from $\mathbf{TW}_+$ by adjoining the rule DL. In the sequel $\mathbf{T'W}_+$ is gentzenized and proved to be decidable.

The Gentzen formulation

The Gentzen formulation of $\mathbf{T'W}_+$ is denoted by $\mathbf{GT'W}_+$.

$X \vdash A_x$ is called a *sequent* provided that if X is nonempty, then any prefix used in X is nonempty.

The *basic* sequents (*axioms*) are all sequents of the form $A_a \vdash A_a$.

The rules of $\mathbf{GT'W}_+$ are: P, W, CL, CR, DL, DR, IL and IR.

Notice that the rules have to be restated such that both the premisses and the conclusion be sequents. In IR the subscript a is called *discharged.*

By a *derivation* in $\mathbf{GT'W}_+$ we mean a tree with usual properties. For a node S of such a tree, the *rank* of S is the number of nodes below S, on the branch to the origin. The *weight* of S is the number of nodes above S, on all branches to which S belongs.

Now we state a theorem to the effect that the concept of a derivation in $\mathbf{GT'W}_+$ is well-defined.

THEOREM 2. *Let $\langle B_1, b_1\rangle, \ldots \langle B_m, b_m\rangle \vdash B_b$ be a node in a derivation of $\langle A_1, a_1\rangle, \ldots, \langle A_n, a_n\rangle \vdash A_a$ in $\mathbf{GT'W}_+$, let $M = \{a_1, \ldots, a_n\}$ and let N be the set of all subscripts discharged in this derivation; then*

(1) $b = b_1 \cup \cdots \cup b_m$;

(2) *if $m \neq \emptyset$, then for all $1 \leq j \leq m$ $b_j \neq \emptyset$;*

(3) *if $a \neq \emptyset$, then $b \neq \emptyset$;*

(4) *if $b \neq \emptyset$, then for any j there are $c_1, \ldots, c_k \in M \cup N$ such that $b_j = c_1 \cup \cdots \cup c_k$.*

PROOF. (1) - (2) are proved by induction on weight; (3) - (4) are proved by induction on rank.

THEOREM 3. *If* (I) $\langle A_1, a_1\rangle, \ldots \langle A_n, a_n\rangle \vdash A_a$ *is derivable in* $\mathbf{GT'W}_+$, *so is* (II) $\langle A_1, a_1'\rangle, \ldots \langle A_n, a_n'\rangle \vdash A_{a'}$, *provided that for any $f, g, f_1, \ldots, f_p \in M$:*

(1) *if $f \subset f_1 \cup \cdots \cup f_p$, then $f' \subset f_1' \cup \cdots \cup f_p'$;*

(2) *if $f = f_1 \cup \cdots \cup f_p$, then $f' = f_1' \cup \cdots \cup f_p'$;*

(3) *if $f \cap g \neq \emptyset$, then $f' \cap g' = \emptyset$;*

(4) *if $\max(f) > \max(g)$, then $\max(f') > \max(g')$;*

(5) *if $\max(f) = \max(g)$, then $\max(f') = \max(g')$.*

PROOF. Let $\mathcal{T}$ be a proof of (I) and let S_k be a node $\langle B_1, b_1\rangle, \ldots, \langle B_m, b_m\rangle \vdash B_b$ of $\mathcal{T}$; by induction on the rank of S_k we shall define a substitution of subscripts $b_1', \ldots, b_m', b'$ for $b_1, \ldots, b_m, b$, respectively, as follows. Let us choose $a_1', \ldots a_n'$ satisfying (1)-(5) and substitute $a_1', \ldots a_n', a'$ for $a_1, \ldots a_n, a$, respectively, where $a' = a_1' \cup \cdots \cup a_n'$, at the origin of $\mathcal{T}$. Suppose that S_k is obtained in $\mathcal{T}$ from S_i

(and S_j) by an application of a rule, and S'_k, obtained from S_k by substitution of $b'_1, \ldots, b'_m, b'$ for $b_1, \ldots, b_m, b$, is defined ($b' = b'_1 \cup \cdots \cup b'_m$). If S_k is obtained by any rule execpt IL and IR, then all subscripts used in S_i (and S_j) are used in S_k; hence, S'_i (and S'_j) is (are) obtained from S_i (and S_j) by substitution of $b'_1, \ldots, b'_m, b'$ for $b_1, \ldots, b_m, b$.

If S_k is obtained by IL from $Z_1 \vdash \langle C, z_1\rangle$ and $Z_2, \langle D, d \cup z_1\rangle \vdash \langle B, d \cup z_1 \cup z_2\rangle$, then Z'_1 and Z'_2 are obtained from Z_1 and Z_2, respectively, by substitution of $b'_1, \ldots, b'_m, b$ for $b_1, \ldots, b_m, b$, respectively. Let z'_1 be the union of all subscripts used in Z'_1; we define $(d \cup z_1)'$: $(d \cup z_1)' = d' \cup z'_1$, where d' is already defined in S'_k.

If S_k is obtained by IR from $Z, \langle C, b_{m+1}\rangle \vdash \langle D, b \cup b_{m+1}\rangle$, where Z is $\langle B_1, b_1\rangle, \ldots, \langle B_m, b_m\rangle$ and B is $C \to D$, then Z' is obtained by substitution of $b'_1, \ldots, b'_m$ for $b_1, \ldots, b_m$. We put: $b'_{m+1} = \{\max(b') + 1\}$, where $b' = b'_1 \cup \cdots \cup b'_m$ and $(b \cup b_{m+1})' = b' \cup b'_{m+1}$.

Let us show that the three $\mathcal{T}'$ just defined is a derivation of (II).

It is easy to verify that (1) - (4) of 2 hold for $\mathcal{T}$.

Let $\mathcal{B}$ be a branch of $\mathcal{T}$ and let $\mathcal{B}'$ be the corresponding branch of $\mathcal{T}'$. It is easy to see that for any subscript c (any c') discharged at a node of $\mathcal{B}$ (of $\mathcal{B}'$) we have: $a_i \cap c = \emptyset$ ($a'_i \cap c' = \emptyset$) for any $1 \leq i \leq n$, and $c \cap d = \emptyset$ ($c' \cap d' = \emptyset$) for any other subscript d discharged at a node of $\mathcal{B}$ (of $\mathcal{B}'$). This suffices to prove

LEMMA 3.1 *For any node* $\langle B_1, b'_1\rangle, \ldots, \langle B, b'_m\rangle \vdash B_{b'}$ *of* $\mathcal{T}'$, *if* $b_j = e_1 \cup \cdots \cup e_q$, *where* $1 \leq j \leq m$ *and* $e_1, \ldots, e_q \in M \cup N$, *then* $b'_j = e'_1 \cup \cdots \cup e'_q$.

To prove the lemma, proceed by an easy induction on the rank of

$$\langle B_1, b_1\rangle, \ldots, \langle B_m, b_m\rangle \vdash B_b.$$

Now we can prove

LEMMA 3.2 *For any* $f, g, f_1, \ldots, f_p \in \{b_1, \ldots, b_m\}$ *we have:*

(1) *if* $f \subset f_1 \cup \cdots \cup f_p$, *then* $f' \subset f'_1 \cup \cdots \cup f'_p$;
(2) *if* $f = f_1 \cup \cdots \cup f_p$, *then* $f' = f'_1 \cup \cdots \cup f'_p$;
(3) *if* $f \cap g = \emptyset$, *then* $f' \cap g' = \emptyset$;
(4) *if* $\max(f) > \max(g)$, *then* $\max(f') > \max(g')$;
(5) *if* $\max(f) = \max(g)$, *then* $\max(f') = \max(g')$.

Details are omitted. Eventually, by using 3.2 and by proceeding inductively on weight, it is easy to show that $\mathcal{T}'$ is a derivation of (II).

As a consequence we have:

THEOREM 4. *If* $\langle A_1, a_1\rangle, \ldots \langle A_n, a_n\rangle \vdash A_a$ *is derivable in* **GT'W$_+$**, *so is* $\langle A_1, a_1 \setminus b\rangle, \ldots \langle A_n, a_n \setminus b\rangle \vdash A_{a \setminus b}$, *provided that for any* $1 \leq i \leq n$

(1) *either* $b \subset a_i$ *or* $a_i \cap b = \emptyset$, *and*
(2) $\max(a_i) > max(b)$.

THEOREM 5. *The following propositions are equivalent:*

(1) $X \vdash A \to B_x$ *is derivable in* **GT'W$_+$**;
(2) $X, A_a \vdash B_{a \cup x}$ *is derivable in* **GT'W$_+$** *for any* a *such that* $a \cap x = \emptyset$, *and* $\max(a) > \max(b)$.

PROOF. (2) $\Rightarrow$ (1) by IR.

(1) $\Rightarrow$ (2) is proved by induction on weight. In the consideration of IR 4 is needed.

Cut elimination theorem for $\mathbf{GT'W_+}$

Let us write X_a instead of X, if a is the only subscript used in X. Also, let us write $X[Y]$ instead of X, if all members of Y are members of X. By $X - Y$ we denote the sequence obtained from X by deleting all members of Y.

For any A, let sf(A) be the set of subformulas of A. A subformula B of A is *proper* if $A \neq B$. The set sf(X) of subformulas of X, where X is $\langle A_1, a_1\rangle, \ldots, \langle A_n, a_n\rangle$, is defined by $\mathrm{sf}(X) = \bigcup_i \mathrm{sf}(A_i)$, $i \in \{1, \ldots, n\}$.

Let us define the *combined degree* cd(X) of X: cd(X) is the total number of occurrences of connectives in sf(X). It is obvious that $\mathrm{cd}(A_a) = \mathrm{cd}(A)$.

The proof of the next lemma is omitted.

LEMMA 6. *For any A, B and X*

(1) *if B is a proper subformula of A, then*
(1.1) $\mathrm{cd}(B) < \mathrm{cd}(A)$;
(1.2) $\mathrm{cd}(X, B) \leq \mathrm{cd}(X, A)$;
(2) $\mathrm{cd}(X, A) = \mathrm{cd}(X, A, A)$;
(3) $\mathrm{cd}(X - Y) \leq \mathrm{cd}(X[Y])$.

Suppose that the sequents $S_1, \ldots, S_n, S_{n+1}$ are derivable with respective weights $w_1, \ldots, w_n, w_{n+1}$; we define the *combined weight* w of $S_1, \ldots, S_n, S_{n+1}$, $w = \max(w_1, \ldots, w_n) + w_{n+1}$. We say that $S_1, \ldots, S_n, S_{n+1}$ are derivable with combined weight w.

THEOREM 7. *If* (1) $X \vdash A_x$ *and* (2) $Y, A_x, Z \vdash B_b$ *are derivable in* $\mathbf{GT'W_+}$ *with combined weight* w*, then for all* Y^* and Z^* (3) $X, Y^*, Z^* \vdash B_b$ *is dervable in* $\mathbf{GT'W_+}$*, where* Y^*, Z^* *is obtained from* Y, Z *by deleting some (possibly none, possibly all) members of the form* A_x.

PROOF. If (1) is basic, then (3) is obtained from (2) by P and C (if (2) and (3) are different). If (2) is basic, then (3) is (1).

If none of (1) and (2) is basic, we proceed by double induction. Our induction hypotheses are:

Hyp 1 The theorem holds for any $A'_{x'}$ of combined degree $\mathrm{cd}(A'_{x'}) < \mathrm{cd}(A_x)$, and any combined weight w;

Hyp 2 The theorem holds for any $A'_{x'}$ of combined degree $\mathrm{cd}(A'_{x'}) = \mathrm{cd}(A_x)$, and any combined weight $w' < w$.

We shall distinguish two cases: (I) the eliminated member A_x has no occurrence in the consequent part of either of the premisses of (1) and no occurrence in the antecedent part of either of the premisses of (2), and (II) otherwise. In (II) there are two subcases: (II.1) A_x occurs in the consequent part of a premiss of (1) and (II.2) otherwise. Furthermore, in (II.2) there are two sub-subcases: (II.2.1) the number of members of the form A_x in all premisses of (2) equals the number of members of the same form in (2), and (II.2.2) otherwise.

Let us consider how (2) could have been obtained.

Suppose that (2) is obtained by IL from (2′) $Z_1 \vdash C_{z_1}$ and (2″) $Z_2, \langle D, a \cup z_1 \rangle \vdash B_d$, where $a \cap z_1 = \emptyset$, $\max(a) < \max(z_1)$, and Y, A_x, Z is $Z_1, Z_2, C \to D_a$. By (1) and 5, $X, \langle A_1, z_1 \rangle \vdash \langle A_2, x \cup z_1 \rangle$ is derivable, where A is $A_1 \to A_2$.

(I) $a = x$ and A is $C \to D$. By (2′), (1′), 6 (1.1) and Hyp 1 we obtain (3′) $Z_1 \vdash \langle A_2, x \cup z_1 \rangle$; by (3′), (2″), 6 (1.1), Hyp 1 and P we obtain (3).

(II.1) The use of premisses of (1), (2), Hyp 2, P and IL is easy.

(II.2.1) By (1), (2′) and Hyp 2 we obtain (3′) $X, Z_1^* \vdash \langle C, z_1 \rangle$; by (1), (2″) and Hyp 2 we derive (3″) $X, Z_2^*, \langle D, a \cup z_1 \rangle \vdash B_b$; hence, (3) is derivable by (3′), (3″), IL, P and C.

(II.2.2) $a = x$ and A is $C \to D$. Since $c \cap z_1 = \emptyset$, the subscript x is not used in Z_1. By (1), (2″) and Hyp 2, (3″) $X, Z_2^*, \langle A_2, x \cup z_1 \rangle \vdash B_b$ is derivable. By (2′), (1′), 6 (1.1) and Hyp 1, (3′) $Z_1, X \vdash \langle A_2, x \cup z_1 \rangle$ is derivable. Hence, by (3′), (3″), 6 (1.1), Hyp 1, P, and C we obtain (3).

The examination of the remaining rules is almost standard and hence omitted. The theorem is proved by double induction.

THEOREM 8. *Suppose that the following conditions are satisfied:*

(a) (1) $\vdash A_1, \ldots,$ (n) $\vdash A_n$ *and* (n+1) $Y[U_a] \vdash B_b$ *are derivable in* **GT'W$_+$** *with combined weight* w;
U_a *and* Y *are* $\langle A_1, a \rangle, \ldots, \langle A_n, a \rangle$ *and* $\langle B_1, b_1 \rangle, \ldots, \langle B_m, b_m \rangle$, *respectively, and* $n \geq 1$;

(c) *all members of* Y *with the subscript* a *are members of* U_a;

(d) *for any* $B_j \leq j \leq m$, *either* $a \subseteq b_j$ *or* $a \cap b_j = \emptyset$;

(e) *if* $a \neq b_j$, *then* $\max(a) < \max(b_j)$;

then (n+1) $Y^* \vdash B_{b \setminus a}$ *is derivable in* **GT'W$_+$**, *where* Y^* *is obtained from* $Y - U$ *by substitution of* $b_j \setminus a$ *for* b_j, *for any* b_j *used in* $Y - U$.

PROOF. If (n+1) is basic, then $n = 1$, $a = b$, A_1 is B and $\vdash B$ is derivable by (1).

If (n+1) is not basic, proceed by double induction. Our induction hypotheses are as Hyp 1 and Hyp 2, with U_a instead of A_x.

Let us distinguish two main cases: (I) there is a $1 \leq i \leq n$ such that A_i is the principal member in the antecedent part of (n+1) (i.e. introduced in the antecedent part of (n+1) by an application of a rule) and (II) otherwise. In (I) there are two subcases: (I.1) no member of U_a occurs in the antecedent part of a premiss of (n+1) and (I.2) otherwise. It is clear that in (I.1) $n = 1$.

Let us consider how (n+1) could have been obtained.

Suppose that (n+1) is obtained by IL from (n+1′) $Z_1 \vdash \langle C, z_1 \rangle$ and (n+1″) $Z_2, \langle D, c \cup z_1 \rangle \vdash B_b$, where $c \cap z_1 = \emptyset$, $\max(c) < \max(z_1)$, and $Y[U_a]$ is $Z_1, Z_2, C \to D_c$. By 5, (i′) $\langle A_i', z_1 \rangle \vdash \langle A_i'', z_1 \rangle$ is derivable, where A_i is $A_i' \to A_i''$.

(I) $a = c$ and A_i is $C \to D$.

(I.1) $n = 1$. By (2′), (1′) and 7, (3′) $Z_1 \vdash \langle A_1'', z_1\rangle$ is derivable. But a is not used in Z_2, z_1 is nonempty, $a \cap z_1 = \emptyset$ and $\max(a) < \max(b_j)$ for any b_j used in Z_2. Hence, the conditions of 4 are satisfied and (3″) $Z_2^*, \langle A_1'', z_1\rangle \vdash B_{b\setminus a}$ is derivable. Hence, by (3′), (3″), 6 (1.1) and Hyp 1, (3) is derivable.

(I.2) Since $a = c$ and $c \cap z_1 = \emptyset$, a is not used in Z_1. By (n+1′), (i′) and 7, (n+2′) $Z_1 \vdash \langle A_i'', z_1\rangle$ is derivable. Let V_a be the sequence of members of U_a which are members of Z_2. Since U is $U[V, \langle A_i, a\rangle]$, by 6 (3) $\mathrm{cd}(V_a) \leq \mathrm{cd}(U_a)$. If $\langle A_i, a\rangle$ is a member of Z_2, then $\mathrm{cd}(U_a) = \mathrm{cd}(V_a)$ by 6 (2); by (1) - (n), (n+1″) and Hyp 2, (n+2″) $Z_2^*, \langle A_i'', z_1\rangle \vdash B_{b\setminus a}$ is derivable. If $\langle A_i, a\rangle$ is not a member of Z_2, then $\mathrm{cd}(V_a) \leq \mathrm{cd}(U_a)$ by 6 (3); by (1) - (i-1), (i+1) - (n), (n+1″) and either Hyp 1 or Hyp 2 we obtain (n+2″) as above. By (n+2′), (n+2″) and 7 we prove (n+2).

(II) Since $a \neq c$, either $a \subset c$ or $a \cap c \neq \emptyset$ by (d), and $\max(a) < \max(c)$ by (e). Let V_a and W_a be subsequences of U_a which are members of Z_1 and Z_2, respectively. By 6 (3) we have both $\mathrm{cd}(V_a) \leq \mathrm{cd}(U_a)$ and $\mathrm{cd}(W_a) \leq \mathrm{cd}(U_a)$. Without loss of generality we may assume that none of V_a and W_a is empty. By (1) - (n), (n+1′) and either Hyp 1 or Hyp 2, (n+2′) $Z_1^* \vdash \langle C, z_1 \setminus a\rangle$ is derivable. Also, by (1) - (n), (n+1″) and either Hyp 1 or Hyp 2, (n+2″) $Z_2^*, \langle D, (c \cup z_1) \setminus a\rangle \vdash B_{b\setminus a}$ is derivable. Hence, by (n+2′), (n+2″) and IL (n+2) is derivable.

Suppose that (n+1) is obtained by CL from (n+1′) $Z_1, C_c \vdash B_b$.

(I) $a = c$ and A_i is $A_i'\&A_i''$, i.e. $C\&D$.

(I.1) Since (1) is not basic, (1′) $\vdash A_1'$ is derivable. By (1′), (2′), 6 (1.1) and Hyp 1, (3) is derivable.

(I.2) Let V_a be the sequence of members of U_a which are members of Z_1. If $\langle A_i, a\rangle$ is a member of Z_1, then $\mathrm{cd}(V_a, \langle A_i', a\rangle) = \mathrm{cd}(U_a)$; if $\langle A_i, a\rangle$ is not a member of Z_1, then $\mathrm{cd}(V_a, \langle A_i', a\rangle) \leq \mathrm{cd}(U_a)$, by 6 (1.2). Now if $\mathrm{cd}(V_a, \langle A_i', a\rangle) = \mathrm{cd}(U_a)$, then by (1) - (i-1), (i′), (i+1) - (n), (n+1′) and Hyp 2 (n+2) is derivable. If $\mathrm{cd}(V_a, \langle A_i', a\rangle) < \mathrm{cd}(U_a)$, then by (1) - (i-1), (i′), (i+1) - (n), (n+1′) and Hyp 1 (n+2) is derivable.

(II) Since $a \neq c$, either $a \subset c$ or $a \cap c = \emptyset$ by (d), and $\max(a) < \max(c)$ by (e). Let V_a be as in (I.2). By (1) - (n), (n+1′) and Hyp 2 (n+2′) $Z_1^*, \langle C, c \setminus a\rangle \vdash B_{b\setminus a}$ is derivable. Hence, (n+2) follows by (n+2′) and CL.

Suppose that (n+1) is obtained by DL from (n+1′) $Z_1, C_c \vdash B_b$ and (n+1″) $Z_1, D_c \vdash B_b$.

(I) $a = c$ and A_i is $C \vee D$, i.e. $A_i' \vee A_i''$. Since (i) is not basic, it is obtained by DL from, say, (i′) $\vdash A_i'$.

(I.1) $n = 1$. By (1′), (2′), 6 (1.1) and Hyp 1, (3) is derivable.

(I.2) Let V_a be the sequence of members of U_a which are members of Z_1. If $\mathrm{cd}(V_a, \langle A_1', a\rangle) = \mathrm{cd}(U_a)$ by 6 (1.2), then by (1) - (i-1), (i′), (i+1) - (n), (n+1′) and Hyp 2 we obtain (n+2). If $\mathrm{cd}(V_a, \langle A_1', a\rangle) < \mathrm{cd}(U_a)$, then by (1) - (i-1), (i′), (i+1) - (n), (n+1′) and Hyp 1 we obtain (n+2).

(II) Since $a \neq c$, either $a \subset c$ or $a \cap c = \emptyset$ by (d), and $\max(a) < \max(c)$ by (e). By (1) - (n), (n+1′) and Hyp 2, (n+2′) $Z_1^*, \langle C, c \setminus a\rangle \vdash B_{b\setminus a}$ is derivable; by (1) -(n), (n+1″) and Hyp 2, (n+2″) $Z_1^*, \langle D, c \setminus a\rangle \vdash B_{b\setminus a}$ is derivable; hence, by (n+2′), (n+2″) and DL we prove (n+2).

Suppose that (n+1) is obtained by W from (n+1″) $Z_1 \vdash B_b$, where $Y[U_a]$ is Z_1, C_c and $c \subseteq b$.

(I) $a = c$ and A_i is C.

(I.1) $n = 1$. Since A is not used in Z_1, for any b_j used in Z_1 either $a \subset b_j$ or $a \cap b_j = \emptyset$, and $\max(a) < \max(b_j)$. Hence, (3) is obtained from (2′) by 4.

(I.2) Let V_a be the sequence of members of U_a which are members of Z_1. By 6 (3), $\mathrm{cd}(V_a) \leq \mathrm{cd}(U_a)$. Hence, (n+2) is obtained by (1) - (n), (n+1) and either Hyp1 or Hyp 2.

(II) Since $a \neq c$, Z_1 is $Z_1[U_a]$, and for any b_j used in Z_1 either $a \subset c$ or $a \cap c = \emptyset$, and $\max(a) < \max(c)$. By (1) - (n), (n+1′) and Hyp 2, (n+2′) $Z_1^* \vdash B_{b\setminus a}$ is derivable. Now (n+2) is obtained by W from (n+2′).

Suppose that (n+1) is obtained by C from (n+1′) $Z_1, C_c, C_c \vdash B_b$, where $Y[U_a]$ is Z_1, C_c.

(I) $a = c$ and A_i is C.

(I.1) $n = 1$. Let us repeat the proof of (1); we may assume that (1), (1) and (2′) are derivable with combined weight $w - 1$ (since so are (1) and (2′)). By (1), (1), (2′), 6 (2) and Hyp 2, $Z_1^* \vdash B_{b\setminus a}$ is derivable.

(I.2) By (1) - (i-1), (i), (i) - (n), (n+1′), 6 (2) and Hyp 2, (n+2) is derivable.

(II) Since $a \neq c$, either $a \subset c$ or $a \cap c = \emptyset$, and $\max(a) < \max(c)$. By (1) - (n), (n+1′) and Hyp 2, (n+2″) $Z_1^*, C_{c\setminus a}, C_{c\setminus a} \vdash B_{b\setminus a}$ is derivable. Hence, (n+2) is obtained from (n+1″) by C.

If (n+1) is obtained by any of the remaining rules, the examination is trivial and hence omitted.

The proof of the theorem is completed by double induction.

THEOREM 9. **GT'W**$_+$ *is closed under* MP.

PROOF. Suppose that (1) $X \vdash A_x$ and (2) $\vdash A \to B_y$ are derivable, where $x \cap y = \emptyset$, and either $\max(x) > \max(y)$ or $y = \emptyset$. By 5, (2′) $Y, A_x \vdash B_{x\cup y}$ is derivable. If $x \neq \emptyset$, then $X, Y \vdash B_{x\cup y}$ follows by 7. If $x = \emptyset$ (and $y = \emptyset$), then $\vdash B$ follows from (1) and (2) by 8.

As a corollary we have

THEOREM 10. $X \vdash A_x$ *in* **T'W**$_+$ *iff* $X \vdash A_x$ *is derivable in* **GT'W**$_+$.

PROOF. Let the reader prove: if A is an axiom of **T'W**$_+$, then $\vdash A$ is derivable in **GT'W**$_+$. The theorem then follows by 1, by definition of **T'W**$_+$, 7 and 9.

Decidability

Let us call a sequent *reduced* iff each of its members occurs at most twice in it. A derivation is reduced iff each of its nodes is reduced.

A branch $\mathcal{B}$ of a derivation tree is *without repetitions* iff any sequent occurs in $\mathcal{B}$ at most once. A derivation $\mathcal{T}$ is without repetitions iff every branch of $\mathcal{T}$ is without repetitions.

The proofs of the following two theorems are omitted.

THEOREM 11. *A reduced sequent S is derivable in* GT'W$_+$ *iff there is a reduced derivation of S.*

THEOREM 12. *For any derivable sequent there is a derivation without repetitions.*

By a *derivation search tree $\mathcal{T}$* for S we mean a tree with S at the origin such that for any node S_k of rank r in $\mathcal{T}$:

(1) there is (are) node(s) S_i (and S_j) of rank $r+1$ such that S_k can be obtained from S_i (and S_j) by an application of a rule or else S_k is basic or else S_k is of the form $p_a \vdash q_a$, where p and q are distinct variables;

(2) S_k is reduced;

(3) $\mathcal{T}$ is without repetitions.

Let $\langle C_1, c_1\rangle, \ldots, \langle C_p, c_p\rangle \vdash \langle C_{p+1}, c_{p+1}\rangle$ and $\langle D_1, q_1\rangle, \ldots, \langle D_q, q_q\rangle \vdash \langle D_{q+1}, q_{q+1}\rangle$ be a premiss S_i and the conclusion S_k, respectively, of a rule ϱ; it is easy to see that for any member A_a of S_i there is a member B_b of S_k such that A_a is a subformula of B_b. Since S_i and S_k can be considered as an ordered $p+1$-tuple and an ordered $q+1$-tuple (if we disregard $\vdash$), there is a function f mapping S_i in S_k and each member A_a of S_i to a member B_b of S_k showing what happens to A_a when ϱ is applied. Let us define f.

P We have: $p = q$ and there is a $1 \le v \le p$ such that $\langle C_v, c_v\rangle = \langle D_{v+1}, d_{v+1}\rangle$ and $\langle C_{v+1}, c_{v+1}\rangle = \langle D_v, d_v\rangle$. If either $u < v$ or $u > v+1$, then $f(\langle C_u, c_u\rangle) = \langle D_u, d_u\rangle$; if $u = v$, then $f(\langle C_u, c_u\rangle) = \langle D_{v+1}, d_{v+1}\rangle$; if $u = v+1$, then $f(\langle C_u, c_u\rangle) = \langle D_v, d_v\rangle$.

W Obviously, $q = p+1$. If $u \le p$, then $f(\langle C_u, c_u\rangle) = \langle D_u, d_u\rangle$; if $u = p+1$, then $f(\langle C_u, c_u\rangle) = \langle D_{q+1}, d_{q+1}\rangle)$.

C Obviously, $p = q+1$. If $u \le p-1$, then $f(\langle C_u, c_u\rangle) = \langle D_u, d_u\rangle$; if $u = p$, then $f(\langle C_u, c_u\rangle) = \langle D_q, d_q\rangle$ and $f(\langle C_{p+1}, c_{p+1}\rangle) = \langle D_{q+1}, d_{q+1}\rangle$.

IL If S_i is the left premiss and $u \le p$, then $f(\langle C_u, c_u\rangle) = \langle D_u, d_u\rangle$; if $u = q+1$, then $f(\langle C_u, c_u\rangle)$ is the principal member of S_k. If S_i is the right premiss and either $u < p$ or $u = p+1$, then $f(\langle C_u, c_u\rangle) = \langle D_{u+v}, d_{u+v}\rangle$, where v is the number of members of the antecedent part of the left premiss. If $u = p$, then $f(\langle C_u, c_u\rangle)$ is the principal member of S_k.

IR Obviously, $p = q+1$. If $u \le p-1$, then $f(\langle C_u, c_u\rangle) = \langle D_u, d_u\rangle$; if either $u = p$ or $u = p+1$, then $f(\langle C_u, c_u\rangle)$ is the principal member of S_k.

If ϱ is any of CL, CR, DL, or DR, then we have $p = q$; we put $f(\langle C_u, c_u\rangle) = \langle D_u, d_u\rangle$.

In the sequel $\mathcal{T}$ is a derivation search tree for S and $\mathcal{B}$ is a branch of $\mathcal{T}$.

For any node S_i of $\mathcal{B}$ of rank r_i and any member A_a of S_i, let us define $f_{i,n}A_a$ as follows, where $0 \le n \le r_i$: $f_{i,0}(A_a) = A_a$; $f_{i,m+1}(A_a) = f(f_{i,m}(A_a))$ for $0 < m < n$. Thus, $f_{i,n}(A_a)$ shows what has happened to A_a after n applications of some rules, the first being an application to S_i.

THEOREM 13. *For any S_i and S_k of $\mathcal{B}$, of ranks r_i and r_k, respectively, $r_i \ge r_k$, and any member of A_a of S_i*

(a) *there is exactly one member B_b of S_k such that* $f_{i,n}(A_a) = B_b$ *and* $n = r_i - r_k$;

(b) *A is a subformula of B;*

(c) *if $A = B$, then $a = b$.*

PROOF. (a) By definition, f is a function. If $r_i = r_k$, then $f_{i,0}(A_a) = A_a$; but S_i is an ordered $p+1$-tuple; hence, A_a is uniquely determined by its position in S_i.

(b) By induction on n in $f_{i,n}$. If $n = 0$, (b) is trivial. Suppose that S_k is obtained by a rule ϱ and S_j is a node of $\mathcal{B}$ which is a premiss in this application of ϱ. By induction hypothesis, $A \in sf(C)$ such that $f_{i,n-1}(A_a) = C_c$ for some C and $n - 1 = r_i - r_j$. Obviously, $f(C_c) = B_b$. By inspection of the rules, we see that $C \in sf(B)$; hemce, $sf(C) \subseteq sf(B)$ and thus $A \in sf(B)$.

(c) By induction on n in $f_{i,n}$. The claim is trivial if $n = 0$. Suppose that S_k is obtained by an application of a rule ϱ and that S_j is a node of $\mathcal{B}$ which is a premiss in this application of ϱ. By induction hypothesis, if $A = C$, where $f_{i,n-1}(A_a) = C_c$, then $a = c$. By inspection of the rules, we see that if $f(C_c) = B_b$ and $B = C$, then $b = c$. Since $A = B$ and $f_{i,n}(A_a) = B_b$, we have $f(f_{i,n-1}(A_a)) = A_b$ and $f(C_c) = A_b$. If A_b is the principal member of S_k, then, by inspection of the rules, we see that C is a proper subformula of A. By (b), A is a subformula of C. Therefore, A_b is not the principal member of S_k. By inspection of the rules, we see that $C_c = B_b = A_b$. Hence $A = C$ and by induction hypothesis $a = c$. Also, we have $f(C_c) = B_b$ and $B = C$; therefore, $a = b = c$.

THEOREM 14. *For any S_i and S_k in $\mathcal{B}$, of ranks r_i and r_k, respectively, $r_i \geq r_k$, and for any A_a and B_b in S_i, if $a \cap b = \emptyset$, $f_{i,n}(A_a) \neq f_{i,n}(B_b)$ and both $f_{i,n}(A_a)$ and $f_{i,n}(B_b)$ are in the antecedent part of S_k, $n = r_i - r_k$, then the subscripts of $f_{i,n}(A_a)$ and $f_{i,n}(B_b)$ are disjoint.*

PROOF. Suppose that the conditions of the theorem are satisfied. If $r_i = r_k$, the theorem is trivial.

Suppose that S_j is a premiss of a rule ϱ such that S_k is the conclusion of ϱ and that S_j is the node of $\mathcal{B}$ of rank r_{k+1}. Furthermore, suppose that $f_{i,n-1}(A_a) = C_c$, $f_{i,n-1}(B_b) = D_d$, $f_{i,n}(A_a) = E_e$ and $f_{i,n}(B_b) = G_g$, and that if C_c and D_d are in the antecedent part of S_j, $C_c \neq D_d$, then $c \cap d = \emptyset$ (induction hypothesis).

If either E_e and G_g are parametric members of S_k or ϱ is either C, P or W, then $E_e = f_{i,n}(A_a) = f(f_{i,n-1}(A_a)) = f(C_c) = C_c$ and $G_g = f_{i,n}(B_b) = f(f_{i,n-1}(B_b)) = f(D_d) = D_d$. Hence, if $E_e \neq G_g$, then $e \cap g = \emptyset$ by induction hypothesis.

Let E_e be a parametric and G_g the principal member of S_k; if ϱ is either CL or DL, then $E_e = f(f_{i,n}(A_a) = f(C_c) = C_c$ and $G_g = f(f_{i,n}(B_b)) = f(D_d)$. It is clear that $c = e$ and $d = g$. Since E_e and G_g are not the same member of S_k, C_c and D_d are not the same member of S_j, by 13 (a). Hence, $e \cap g = c \cap d = \emptyset$ by induction hypothesis.

Let ϱ be IL. If S_j is the left premiss, then $e \cap g = \emptyset$ by definition of ϱ. If S_j is the right premiss, then $E_e = f(f_{i,n-1}(A_a)) = f(C_c) = C_c$, $G_g = f(f_{i,n-1}(B_b)) = f(D_d)$ and $g \subset d$, by definition of ϱ. Since E_e and G_g are two distinct members of S_k, C_c

and D_d are two distinct members of S_j; hence, $c \cap d = \emptyset$ by induction hypothesis and thus $e \cap g = \emptyset$.

THEOREM 15. *Let $S_1, \dots, S_n$ be nodes of $\mathcal{B}$ of ranks $r_1, \dots, r_n$, respectively, let $\langle A, a_1 \rangle, \dots, \langle A, a_n$ belong to the antecedent parts of $S_1, \dots, S_n$, respectively, and the subscripts $a_1, \dots, a_n$ be pairwise disjoint. If there is a S_k in $\mathcal{B}$ of rank r_k, $r_k \leq r_n$, and a member B_b of S_k such that for all $1 \leq i \leq n$, $m_i = r_k - r_i$, $f_{i,m_i}(\langle A, a_i \rangle) = B_b$, then A occurs $p \geq n$ times as a subformula in B.*

PROOF. If $n = 1$, the theorem holds by 13 (b). Suppose that $n > 1$, that the conditions of the theorem are satisfied and that S_k is the first node of $\mathcal{B}$ after S_n with such a property. Let us consider how S_k could have been obtained.

Suppose that S_k is obtained by an application of a rule ϱ and that S_j is a premiss of ϱ and a node of $\mathcal{B}$ of rank $r_k + 1$. It is clear that ϱ is neither P nor W. By inspection of the rules, we see that for any member D_d of S_k there are at most two members E_e and G_g in the premiss(es) of S_k, in the application of ϱ, such that $f(E_e) = f(G_g) = D_d$. Since all $\langle A, a_1 \rangle, \dots, \langle A, a_n \rangle$ occur in the nodes of $\mathcal{B}$, there is a member C_c in S_j such that, say, $f_{i,m_i-1}(\langle A, a_i \rangle) = C_c$ for all $1 \leq i \leq n$, and there is a member $\langle D', d' \rangle$ of S_j such that $f_{n,m_n-1}(\langle A, a_n \rangle) = \langle D', d' \rangle$, where $D_d = f(C_c) = f(\langle D', d' \rangle$, and C_c and $\langle D', d' \rangle$ are distinct members of S_j. If both C_c and $\langle D', d' \rangle$ are in the antecedent part of S_j, then $c \cap d' = \emptyset$ by 14. Hence, ϱ is not C. By inspection of the rules, we eliminate all remaining ones as candidates for ϱ, except IR. Furthermore, by induction hypothesis, A occurs $p' \geq n - 1$ times as a subformula of C.

Let S_k be $X \vdash D_d$ obtained from $X, E_e \vdash G_{d \cup e}$ by IR, where $D = E \rightarrow G$, and either $E_e = C_c$ and $G_{d \cup e} = \langle D', d' \rangle$ or else $E_e = \langle D', d' \rangle$ and $G_{d \cup e} = C_c$. By 13, A occurs p'' times as a subformula in D', $p'' \geq 1$. By using induction hypothesis and the fact that $D = E \rightarrow G$, we conclude that A occurs $p' + p'' \geq n$ times as a subformula in D.

Suppose that S_k is not the first member of $\mathcal{B}$ of rank r_k, $r_k \leq r_n$, such that S_k contains D_d; then the theorem follows by 13.

This completes the proof of the theorem.

Let ϱ be a rule; for a member S of $\mathcal{B}$ we say that S is *ϱ-member* iff it is a premiss in an application of ϱ.

THEOREM 16. *If the number of* IR*-members is finite, then $\mathcal{B}$ is finite.*

PROOF. Suppose that the number of IR-members is finite; then there is a finite number of subscripts disharged in $\mathcal{B}$. By 2 (4), each subscript used in $\mathcal{B}$ is the union of some subscripts used at the origin of $\mathcal{T}$ and some other subscripts discharged in $\mathcal{B}$. Hence, the number of subscripts used in the nodes of $\mathcal{B}$ is finite. By 13 it follows that every member S_k of $\mathcal{B}$consists of subscripted subformulas of the members of S. Therefore, there is a finite number of reduced sequents that can be constructed. Since $\mathcal{B}$ is without repetitions, it is finite.

THEOREM 17. *Every derivation search tree $\mathcal{T}$ is finite.*

PROOF. Suppose that $\mathcal{T}$ is infinite; by König's lemma, $\mathcal{T}$ has an infinite branch $\mathcal{B}$. By 16, there is an infinite number of IR-members of $\mathcal{B}$. Hence, an infinite number of pairwise disjoint subscripts is used in $\mathcal{B}$(those discharged in $\mathcal{B}$). Since the number of subformulas of members of S is finite, there is a formula A such that A is a subformula of a member D_d of S and A occurs in the nodes of $\mathcal{B}$ with an infinite number of pairwise disjoint subscripts $a_1, \ldots, a_n, \ldots$. By 15, for any S_k of $\mathcal{B}$, any $\langle D', d' \rangle$ of S_k, any finite n and any $a_1, \ldots, a_n$, if $f_{i,m_i}(\langle A, a_i \rangle) = \langle D', d' \rangle$ for any $1 \leq i \leq n$, then A occurs p times as a subformula in D', $p' \geq n$. It is clear that S is such a S_k of $\mathcal{B}$ and D_d is a $\langle D', d' \rangle$ in S. Hence, A occurs at least n times as a subformula in D, for any finite n. This is absurd. Hence, every branch $\mathcal{B}$ of $\mathcal{T}$ is finite and thus $\mathcal{T}$ is finite.

THEOREM 18. GT'W$_+$ *is decidable.*

PROOF. By 5, IR can be used in the following form:

IR$'$ From $X, \{\max(x)+1\}A \vdash \langle B, x \cup \{\max(x)+1\} \rangle$ to infer $X \vdash A \rightarrow B_x$.

By 17, it follows that the number of derivation search trees for a given S is finite.

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