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INCIDENCE STRUCTURES WITH n-METRICS

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ABSTRACT. *In this paper we define and examine structures* $(V, B, \in, d)$, *where* $(V, B, \in)$ *is a t-design or* $[n, n+m]$*-net,* d *is a mapping from* V^t *to* $R\backslash R^-$ *which satisfies certain axioms analogous to the axioms of usual metrics. The mapping* d *is a generalization of the usual metrics so we call it n-metric function and we call the structure* $(V, B, \in, d)$ *n-metric space. Then we define a t-induced topology and examine it.*

An *incidence structure* is a triple $D = (V, B, I)$, where V and B are disjoint sets and $I \subseteq V \times B$. The elements of V will be called *points*, those of B blocks. If A is any point, (A) will denote the set of blocks incident with A, i.e.

$$(A) = \{b \in B \mid A \ I \ b\} ,$$

and more generally for any subset $\{A_1, A_2, \dots, A_n\}$ of point set

$$(A_1, A_2, \dots, A_n) = \{b \in B \mid A_i \ I \ b \text{ for each } i = 1, \dots, n\} .$$

Similarly we write

$$(b_1, b_2, \dots, b_n) = \{A \in V \mid A \ I \ b_i \text{ for each } i = 1, \dots, n\} .$$

We will consider incidence structures where distinct blocks have distinct point sets. We can identify each block b with the corresponding point set (b) and the incidence relation I with the membership relation.

DEFINITION 1. *A finite incidence structure* $D = (V, B, I)$ *is called a block design with parameters* v, k, λ $(v, k, \lambda \in N)$ *if it satisfies the following conditions:*

a) $|V| = v$;

b) $|(P, Q)| = \lambda$ *for all* $\{P, Q\} \subseteq V$, *i.e. any two distinct points are joined by exactly* λ *blocks;*

c) $|(b)| = k$ *for any block* b.

A finite incidence structure $D = (V, B, I)$ *is called t-design* $(S_\lambda(t, k, v))$ *if it satisfies conditions* $a), c)$ *and* $b')$, *where*

$b')$ *any* t *distinct point are joined by exactly* λ *blocks. In the case* $\lambda = 1$, *a t-design is called a Steiner system* $S(t, k, v)$. *An* $S(3, 4, v)$ *is called a*

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Steiner quadruple system.

The following result is proved in [1].

THEOREM 1. *Let D be a t-design and let $s \leq t$ be a positive integer. Then D is also an s-design. If D has parameters v, k and λ_t (where λ_t is the number of blocks through a t-set), then the parameter λ_s (the number of blocks through an s-set) is given by*

$$\lambda_s = \lambda_t \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s} .$$

DEFINITION 2. *Let V and B be nonempty sets, let $B = B_1 \cup \cdots \cup B_{n+m}$ be a partition of B, where $n \geq 2$, $m \geq 1$, and let $I \subseteq V \times B$. The structure (V, B, I) is called an $[n, m+n]$-net if the following conditions are satisfied:*

i) if $P \in V$, then there exists exactly one sequence $b_1, b_2, \dots, b_{n+m} \in B$ such that $P\ I\ b_s$, $b_s \in B_s$, for all $s \in N_{n+m}$;

ii) if $\varphi : N_n \to N_{n+m}$ is an injection and $b_s \in B_{\varphi(s)}$, then there exists exactly one $P \in V$ such that $P\ I\ b_s$ for all $s \in N_n$.

The sets $B_1, B_2, \dots, B_{n+m}$ are called classes of parallel blocks.

The following result is proved in [2].

THEOREM 2. *A) The classes of parallel blocks have the same cardinality $|B_i| = q$, $i = 1, \dots, n+m$.*

b) If $b_1, b_2, \dots, b_r$ are r-blocks from r-diferent classes where $1 \leq r \leq n$ then $|(b_1, b_2, \dots, b_r)| = q^{n-r}$.

In the following definition V may be any nonempty set, but we take V to be the set of points or the set of blocks of an incidence structure.

DEFINITION 3. *Let $D = (V, B, \in)$ be an incidence structure. The mapping $d : V^t \to R \setminus R^-$ ($B^t \to R \setminus R^-$) is called t-metric function if it satisfies the following axioms:*

i) $d(A_1, A_2, \dots, A_t) = 0 \iff A_1 = A_2 = \cdots = A_t$;

ii) $d(A_1, A_2, \dots, A_t) = d(A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_t})$, for every permutation α on t elements, $\alpha \in \Sigma_t$;

iii) $d(A_1, \dots, A_{t-2}, X, Y) = d(A_1, \dots, A_{t-2}, X, Z) + d(A_1, \dots, A_{t-2}, Z, Y)$.

For $n = 2$, d is the usual metric. *We call the structure $D = (V, B, \in, d)$, t-metric space.*

THEOREM 3. *Let $D = (V, B, \in)$ be an $[n, m+n]$-net. The mapping $d : B^t \to R \setminus R^-$, $2 \leq t \leq n$ defined by:*

$$d(b_1, b_2, \dots, b_t) = |(b_i)| - |(b_1, b_2, \dots, b_t)| ,$$

for every $(b_1^t) \in B^t$, $i = 1, \dots, t$, is a t-metric function.

Note. (b_1^t) denotes the sequences $(b_1, b_2, \dots, b_t)$.

PROOF. For any block $b \in B$, $|(b)| = q^{n-1}$. So, it follows that d is a well defined mapping. If $b_1 = b_2 = \ldots = b_t$, then $d(b_1, b_2, \ldots, b_t) = |(b_i)| - |(b_i)| = 0$. If $b_i \neq b_j$, then

$$|(b_i)| > |(b_1, b_2, \ldots, b_t)|, \; d(b_1, b_2, \ldots, b_t) = |(b_i)| - |(b_1, b_2, \ldots, b_t)| > 0 ,$$

and condition *i*) holds. The condition *ii*) is obvious so it remains to prove the condition *iii*). If $b_i \parallel b_j$ for some $i \neq j$, then

$$d(b_1, \ldots, b_{t-1}, x) = d(b_1, \ldots, b_{t-1}, y) = |(b_i)| ,$$

and *iii*) holds. Bacause of the above we may suppose that $b_i \nparallel b_j$, for every $i \neq j$, $i, j = 1, \ldots, t$. If $x = y$ then

$$d(b_1, \ldots, b_{t-1}, x) = d(b_1, \ldots, b_{t-1}, y) .$$

If $x \parallel y$ then $d(b_1, \ldots, b_{t-2}, x, y) = |(b_i)|$ so that

$$\begin{aligned} d(b_1, \ldots, b_{t-1}, x) &= |(b_i)| - |(b_1, \ldots, b_{t-1}, x)| \le |(b_i)| = d(b_1, \ldots, b_{t-2}, x, y) \\ &\le d(b_1, \ldots, b_{t-2}, x, y) + d(b_1, \ldots, b_{t-1}, y) . \end{aligned}$$

In both cases the condition *iii*) holds. We may suppose that $x \neq y$ and $x \nparallel y$. If $y \parallel b_i$ for some $i = 1, \ldots, t-1$, then

$$\begin{aligned} d(b_1, \ldots, b_{t-1}, y) &= |(b_i)| - d(b_1, \ldots, b_{t-1}, x) \le |(b_i)| = d(b_1, \ldots, b_{t-1}, y) \\ &\le d(b_1, \ldots, b_{t-1}, y) + d(b_1, \ldots, b_{t-2}, x, y) . \end{aligned}$$

In the case, when $x \parallel b_i$ for some $i = 1, \ldots, t-1$, and $y \nparallel b_i$ for every $i = 1, \ldots, t-1$, we have

$$\begin{aligned} d(b_1 \ldots, b_{t-1}, y) \ge q^{n-1} - q^{n-2}, \; & d(b_1, \ldots, b_{t-2}, x, y) \ge q^{n-1} - q^{n-2}, \\ d(b_1, \ldots, b_{t-1}, x) &= q^{n-1} \le 2(q^{n-1} - q^{n-2}) \\ &\le d(b_1, \ldots, b_{t-1}, y) + d(b_1, \ldots, b_{t-2}, x, y) . \end{aligned}$$

So we are left with the case $x \nparallel b_i$, $y \nparallel b_i$ for every $i = 1, \ldots, t-1$. If $x = b_i$ for some $i = 1, \ldots, t-1$, then

$$\begin{aligned} d(b_1, \ldots, b_{t-1}, x) &\le q^{n-1} - q^{n-t+1} \le 2(q^{n-1} - q^{n-2}) \\ &\le d(b_1, \ldots, b_{t-1}, y) + d(b_1, \ldots, b_{t-2}, x, y) . \end{aligned}$$

If $x \neq b_i$ for every $i = 1, \ldots, t-1$ then

$$\begin{aligned} d(b_1, \ldots, b_{t-1}, x) &\le q^{n-1} - q^{n-t} \le 2(q^{n-1} - q^{n-2}) \\ &\le d(b_1, \ldots, b_{t-1}, y) + d(b_1, \ldots, b_{t-2}, x, y) . \; \square \end{aligned}$$

From Theorem 3. it follows that the structure $D = (V, B, \in, d)$ where $(V, B, \in)$ is an $[n, n+m]$-net and d is a t-distance function for every $t \in N$, $2 \le t \le n$, is a t-metric space.

The following theorem proves that every t-design $D = (V, B, \in)$ produces an n-metric for every $2 \le n \le t$.

THEOREM 4. *Let* $D = (V, B, \in)$ *be a t-design* $S_\lambda(t, k, v)$. *The mapping* $d : V^n \to R \setminus R^-$ *defined by:*

$$d(A_1, \ldots, A_n) = |(A_i)| - |(A_1, \ldots, A_n)| ,$$

$2 \le n \le t$, $n \in N$, *is an n-metric function.*

PROOF. For any point $A_i \in V$, $|(A_i)| = r$, and for any $n \in N$ with $2 \le n \le t$, $\lambda_n \ge \lambda_t = \lambda$ where $\lambda_n = |(A_1, \ldots, A_n)|$. So it follows that d is a well defined mapping. If $A_1 = A_2 = \cdots = A_n = A$ then

$$d(A_1,\dots,A_n)=d(A,\dots,A)=|(A)|-|(A,\dots,A)|=|(A)|-|(A)|=0\ .$$

On the contrary, if $A_i\neq A_j$ for some $i\neq j$, then $|(A_1,\dots,A_n)|<|(A_i)|$, $d(A_1,\dots,A_n)=|(A_i)|-|(A_1,\dots,A_n)|>0$ and condition $i)$ holds. The condition $ii)$ is obvious so it remains to prove the condition $iii)$. Let $A_1,\dots,A_{n-1},Y,Z\in V$. If $Y=Z$ then

$$d(A_1,\dots,A_{n-1},Y)=d(A_1,\dots,A_{n-1},Z)\ ,\quad d(A_1,\dots,A_{n-2},Y,Z)\geq 0\ ,$$
$$d(A_1,\dots,A_{n-1},Y)\leq d(A_1,\dots,A_{n-2},Z)+d(A_1,\dots,A_{n-2},Y,Z)\ ,$$

and the condition $iii)$ holds. Because of the above, we may suppose that $Y\neq Z$. If $Y=A_i$ for some $i=1,\dots,n-1$, then

$$d(A_1,\dots,A_{i-1},A_{i+1},\dots,A_{n-1},Y,Z)=d(A_1,\dots,A_{n-1},Z)\ ,$$
$$d(A_1,\dots,A_{n-2},Y,Z)\geq 0\ ,$$

and

$$\begin{aligned}d(A_1,\dots,A_{n-1},Y)&\leq d(A_1,\dots,A_{i-1},A_{i+1},\dots,A_{n-1},Y,Z)\\&=d(A_1,\dots,A_{n-1},Z)\\&\leq d(A_1,\dots,A_{n-1},Z)+d(A_1,\dots,A_{n-2},Y,Z)\ .\end{aligned}$$

If $Z=A_i$ for some $i=1,\dots,n-1$ and $Y\neq A_i$ for every $i=1,\dots n-1$, then

$$|(A_1,\dots,A_{n-1},Z)|\geq|(A_1,\dots,A_{n-2},Y,Z)|\ ,$$
$$d(A_1,\dots,A_{n-2},Y,Z)\leq d(A_1,\dots,A_{n-1},Z)\ ,$$

and

$$\begin{aligned}d(A_1,\dots,A_{n-1},Y)&\leq\lambda_1-\lambda_n\leq 2(\lambda_1-\lambda_2)\leq d(A_1,\dots,A_{n-2},Y,Z)\\&\leq d(A_1,\dots,A_{n-2},Y,Z)+d(A_1,\dots,A_{n-1},Z)\ .\end{aligned}$$

In the case $Y\neq Z$, $Y\neq A_i$, $Z\neq A_i$ for every $i=1,\dots,n-1$, we have

$$d(A_1,\dots,A_{n-1},Y)=d(A_1,\dots,A_{n-1},Z)\ ,$$
$$d(A_1,\dots,A_{n-2},Y,Z)>0\ ,$$

and

$$d(A_1,\dots,A_{n-1},Y)\leq d(A_1,\dots,A_{n-1},Z)+d(A_1,\dots,A_{n-2},Y,Z)\ .\ \square$$

DEFINITION 4. *Let $D=(V,B,\in,d)$ be an n-metric space, $(A_1^t)\in V^t$, $t\in N$, $1\leq t<n$, $\epsilon\in R^+$. The subset $B(A_1^{n-t},\epsilon)$ of the set V^t defined by*

$$B(A_1^{n-t},\epsilon)=\{(Y_1^t)\mid(Y_1^t)\in V^t,\ d(A_1^{n-t},Y_1^t)<\epsilon\}$$

is called an open $\epsilon-(n-t,t)$-ball with the center (A_1^{n-t}) and radius ϵ.

DEFINITION 5. *The topology generated by the set of all open $\epsilon-(n-t,t)$-balls, $B(A_1^{n-t},\epsilon)$, is called a topology t-induced by the n-metric function.*

The following example shows that there exists a finite n-metric space so that the t-induced topology is not discrete.

EXAMPLE 1. Let $D=(V,B,\in)$ be a Steiner quadruple system $S(3,4,8)$, $V=\{1,2,3,4,5,6,7,8\}$,

$$B=\left\{\begin{matrix}\{1,2,3,4\}, & \{1,2,5,6\}, & \{1,2,7,8\}, & \{1,3,5,7\}, & \{1,3,6,8\}, & \{1,4,5,8\}, & \{1,4,6,7\}\\ \{5,6,7,8\}, & \{3,4,7,8\}, & \{3,4,5,6\}, & \{2,4,6,8\}, & \{2,4,5,7\}, & \{2,3,6,7\}, & \{2,3,5,8\}\end{matrix}\right\}.$$

The 4-metric $d:V^4\to R\backslash R^-$ from Theorem 4. is defined by: for $X\neq Y\neq Z\neq T$,

$$d(X,X,X,X)=0,\ d(X,X,X,Y)=7-3=4,\ d(X,X,Y,Y)=7-4=3,$$

$$d(X,X,Z,Y) = 7-1 = 6,\ d(X,Y,X,Y) = 4,$$

$$d(X,Y,Z,T) = \begin{cases} 7 & \text{if points } X,Y,Z,T \text{ are not incident with the same block} \\ 6 & \text{if points } X,Y,Z,T \text{ are incident with the same block} \end{cases}.$$

We may define open $\epsilon-(3,1)$-balls, open $\epsilon-(2,2)$-balls and open $\epsilon-(1,2)$-balls.

The open $\epsilon-(3,1)$-balls are:

$$B(X,X,X,1) = \{X\},\ B(X,X,Y,5) = \{X,Y\},$$
$$B(X,X,X,5) = V,\ B(X,X,Y,7) = V,$$
$$B(X,Y,Z,7) = \{X,Y,Z,T \mid X,Y,Z,T \text{ are incident with the same block}\}.$$

For every $X \in V$ the subset $\{X\}$ is open, and so, the 1-induced topology is discrete.

The open $\epsilon-(2,2)$-balls are:

$$B(X,X,1) = \{(X,X)\},\ B(X,X,5) = \{(X,Y) \mid Y \in V\},$$
$$B(X,X,7) = V,\ B(X,Y,5) = \{(X,X),(Y,Y),(X,Y)\},$$
$$B(X,Y,7) = \{(X,X),(Y,Y),(X,Z),(Y,Z),(U,T)$$
$$\mid Z \in V,\ X,Y,U,T \text{ are incident with the same block }\}.$$

$B(X,X,5) \cap B(Y,Y,5) = \{(X,Y)\}$. For every $X \neq Y$ the subsets $\{(X,Y)\}$, $X \neq Y$, are open sets, and so the 2-incident topology is discrete.

The open $\epsilon-(1,3)$-balls are: $B(X,1) = \{(X,X,X)\}$,

$$B(X,5) = \{(X,X,X),(Y,Y,Y),(X,X,Y),(X,Y,Y) \mid Y \in V,\ Y \neq X\}$$
$$B(X,7) = \{(X,X,X),(Y,Y,Y),(X,X,Y),(X,Y,Y),(Y,Y,Z),(X,Y,Z)$$
$$\mid Y,Z \in V,\ Y \neq X,\ Z \neq X,\ Z \neq Y\}.$$

The basis $\mathcal{I}$ of the 3-induced topology $\mathcal{O}$ is:

$$\mathcal{I} = \{B(X,1),\ B(X,5),\ B(X,7), \{(U,U,U) \mid U \in V\},$$
$$\{(U,U,U),(X,X,Y),(X,Y,Y) \mid U \in V\},$$
$$\{(U,U,U),(X,X,Y),(X,Y,Y),(W,W,T),(X,Y,T)$$
$$\mid W,U,T \in V, W \neq T, T \neq X, T \neq Y\}\}$$

for all $X,Y \in V$, $X \neq Y$. The subsets $\{(X,Y,Z)\}$, $X \neq Y \neq Z$, are not open and so the 3-induced topology is not discrete.

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