Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 6 (1992), 141–149

FILOMAT–20, Niš, September 26–28, 1991

INFLATIONS OF A BAND OF MONOIDS

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Abstract. *In this paper constructions of inflations of bands of monoids by systems of homomorphisms are given.*

Using the method from [3] we give constructions for inflations of bands of monoids in the general case, and also in some special cases (inflations of Rédei's bands of monoids). Moreover, some characterizations for inflations of bands of groups and for a Rédei's bands of groups are given.

Let T be a subsemigroup of a semigroup S. A homomorphism φ of S onto T is a *retraction* if $\varphi(t) = t$ for all $t \in T$. A semigroup S is an *inflation* of a semigroup T if T is a subsemigroup of S, $S^2 \subseteq T$ and there exists a retraction of S onto T. By G_e we denote the maximal subgroup of a semigroup S with the identity e. A band E is a *Rédei's band* if $ef \in \{e, f\}$ for all $e, f \in E$. Let $\leq$ be a *quasiorder* (i.e. reflexive and transitive binary relation) on a set X. Then $x < y \iff x \leq y \land x \neq y,\ x, y \in X$. Let $S_i,\ i \in I$ be a family of semigroups with pairwise disjoint sets of elements and let $\leq$ be a quasiorder on the index set I. A system φ_{ij} of homomorphisms of S_j into S_i defined for all $i, j \in I$ such that $i \leq j$ we call the *system of homomorphisms over the quasiorder* $\leq$. If, besides that, the following properties hold:

(*i*) *φ_{ii} is the identical automorphism of S_i for every $i \in I$,*

(*ii*) $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ *if* $i < j < k$,

then the system φ_{ij} we call the *transitive system of homomorphisms over* $\leq$. In this paper we will use the following quasiorders defined on a band I:

$$i \leq_1 j \iff ji = i\,, \qquad i \leq_2 j \iff ij = i\,, \qquad i, j \in I\,.$$

If I is a Rédei's band and a relation $\leq_3$ is defined on I by

$$i \leq_3 j \iff i \leq_1 j \lor i \leq_2 j\,, \qquad i, j \in I\,,$$

then by Lemma 1. [8] it follows that $\leq_3$ is also a quasiorder on I.

For undefined notions and notations we refer to [1] and [7].

Supported by Grant 0401A *of Science Fund of Serbia through Math. Inst. SANU*
1991 *Mathematics subject classification. Primary:* 20M10

THEOREM 1. *Let I be a band. To each $i \in I$ we associate a semigroup S_i such that $S_i \cap S_j = \varnothing$ if $i \neq j$. Let S_i be an inflation of a monoid G_i with the identity e_i, $i \in I$. Let φ_{ij} and ψ_{ij} be systems of homomorphisms over $\leq_1$ and $\leq_2$, respectively, for which the following properties hold:*

(1) φ_{ij} *and* ψ_{ij} *are the identical automorphisms of* S_i, *for every* $i \in I$;

(2) $$\varphi_{ij} \circ \varphi_{jk}(s_k) = \varphi_{ik}(s_k)\varphi_{ij}(e_j)\,, \qquad i \leq_1 j \leq_1 k\,,$$

(3) $$\psi_{ij} \circ \psi_{jk}(s_k) = \psi_{ij}(e_j)\psi_{ik}(s_k)\,, \qquad i \leq_2 j \leq_2 k\,.$$

Let (a_{ij}) be an $I \times I$-matrix over $S = \cup\{S_i \mid i \in I\}$ such that $a_{ij} \in S_{ij}$, $a_{ii} = e_i$ and

(4) $$\varphi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k} = a_{i,jk}\psi_{ijk,jk}(\varphi_{jk,j}(s_j)a_{jk})\,,$$

for every $i, j, k \in I$. Define a multiplication $*$ on $S = \cup\{S_i \mid i \in I\}$ by:

(5) $$s_i * s_j = \varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j)\,, \qquad s_i \in S_i,\ s_j \in S_j\,.$$

Then $(S, *)$ is an inflation of a band of monoids.

Conversely, every inflation of a band of monoids can be so constructed.

PROOF. Let the conditions of this Theorem hold. Then by Theorem 1. [3] $(G, *)$ is a band of monoids, where $G = \cup\{G_i \mid i \in I\}$. That $(S, *)$ is a semigroup we prove in a similar way as in the proof of Theorem 1. [3]. For $s_i \in S_i$, $s_j \in S_j$ we have that $s_i * s_j = \varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j) \in S_{ij}S_{ij}S_{ij} \subseteq G_{ij}$, so $S^2 = G$. Let $\phi_i : S_i \to G_i$, $i \in I$, be a retraction. Then for $s_i \in S_i$, $\phi_i(s_i) = e_i s_i = s_i e_i$. Define a mapping $\phi : S \to G$ by:

$$\phi(x) = \phi_i(x) \qquad \text{if } x \in S_i,\ i \in I\,.$$

Let $s_i \in S_i$, $s_j \in S_j$. Then

$$\begin{aligned}
\phi(s_i) * \phi(s_j) &= \phi_i(s_i) * \phi_j(s_j) \\
&= \varphi_{ij,i}(\phi_i(s_i))a_{ij}\psi_{ij,j}(\phi_j(s_j)) \\
&= \varphi_{ij,i}(s_i e_i)a_{ij}\psi_{ij,j}(e_j s_j) \\
&= \varphi_{ij,i}(s_i)\varphi_{ij,i}(e_i)a_{ij}\psi_{ij,j}(e_j)\psi_{ij,j}(s_j) \\
&= \varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j) && \text{by (2) and (3),} \\
&= s_i * s_j = \phi_{ij}((s_i * s_j) && \text{since } s_i * s_j \in G_{ij}, \\
&= \phi(s_i * s_j)\,.
\end{aligned}$$

Therefore, ϕ is a retraction of S onto G, so S is an inflation of a band of monoids G.

Conversely, let S be an inflation of a semigroup G, let G be a band I of monoids G_i, $i \in I$, and let $\phi : S \to G$ be a retraction. Let $S_i = \phi^{-1}(G_i)$, $i \in I$. Then S is a band I of pairwise disjoint semigroups S_i, $i \in I$, and for every $i \in I$, S_i is an inflation of a monoid G_i. Clearly, $\phi(x_i) = e_i x_i = x_i e_i$, $x_i \in S_i$, $i \in I$. Define mappings φ_{ij} and ψ_{ij} of S_j into S_i over $\leq_1$ and $\leq_2$, respectively, by

$$\varphi_{ij}(s_j) = \begin{cases} s_i & \text{if } i = j \\ s_j e_i & \text{if } i \neq j \end{cases}, \qquad \psi_{ij}(s_j) = \begin{cases} s_i & \text{if } i = j \\ e_i s_j & \text{if } i \neq j \end{cases}.$$

Immediately we show that φ_{ij} and ψ_{ij} are homomorphisms. It is clear that $a_{ij} = e_i e_j \in G_{ij}$ and that $a_{ii} = e_i$ for all $i, j \in I$. Using the fact that

$$\varphi_{ij}(s_j) = s_j e_i = \phi(s_j)\phi(e_i) = s_j e_j e_i = \varphi_{ij}(s_j e_j) = \varphi_{ij}(\phi(s_j))\,,$$

and, analogously, that $\psi_{ij}(s_j) = \psi(\phi(s_j))$, then by Theorem 1. [3] we have that the conditions (2),(3) and (4) hold. For $s_i \in S_i$, $s_j \in S_j$ we obtain that

$$\begin{aligned} s_i s_j &= \phi(s_i)\phi(s_j) = \varphi_{ij,i}(\phi(s_i))a_{ij}\psi_{ij,j}(\phi(s_j)) \\ &= \varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j)\ .\ \square \end{aligned}$$

EXAMPLE 1. The semigroup given by the following table

	e_i	x_i	e_j	x_j
e_i	e_i	e_i	e_j	e_j
x_i	e_i	e_i	e_j	e_j
e_j	e_i	e_i	e_j	e_j
x_j	e_i	e_i	e_j	e_j

is an inflation of a band of groups and the multiplication on S is determined by the following homomorphisms

$$\varphi_{ij} = \begin{pmatrix} e_j & x_j \\ e_i & x_i \end{pmatrix}\ ,\quad \psi_{ji} = \begin{pmatrix} e_i & x_i \\ e_j & e_j \end{pmatrix}\ ,\quad (i <_1 j <_1 i)\ ,$$

and $a_{ij} = a_{jj} = e_j$, $a_{ji} = a_{ii} = e_i$. Clearly, this representation of homomorphisms φ_{ij} and ψ_{ji} is different to the representation of its in Theorem 1. Therefore, systems of homomorphisms are not determined uniquely.

By the following theorem we give the construction of an inflation of a band of monoids different to the construction from Theorem 1. This other construction give the connection between systems of homomorphisms and retractions.

THEOREM 2. *Let I be a band. To each $i \in I$ we associate a semigroup S_i such that $S_i \cap S_j = \varnothing$ if $i \neq j$. Let S_i be an inflation of a monoid G_i with the identity e_i, $i \in I$. Let φ_{ij} and ψ_{ij} be systems of homomorphisms over $\leq_1$ and $\leq_2$, respectively, for which the following properties hold:*

(6) *for every $i \in I$, $\varphi_{ij} = \psi_{ij}$ is a retraction of S_i onto G_i;*

(7) $$\varphi_{ii} \circ \varphi_{ij} = \varphi_{ij} \circ \varphi_{jj} = \varphi_{ij}\ ,\qquad i \leq_1 j\ ,$$

(8) $$\psi_{ii} \circ \psi_{ij} = \psi_{ij} \circ \psi_{jj} = \psi_{ij}\ ,\qquad i \leq_2 j\ ,$$

(9) $$\varphi_{ij} \circ \varphi_{jk}(s_k) = \varphi_{ik}(s_k)\varphi_{ij}(e_j)\ ,\qquad i <_1 j <_1 k\ ,$$

(10) $$\psi_{ij} \circ \psi_{jk}(s_k) = \psi_{ij}(e_j)\psi_{ik}(s_k)\ ,\qquad i <_2 j <_2 k\ .$$

Let (a_{ij}) be an $I \times I$-matrix over $S = \cup\{S_i \mid i \in I\}$ such that $a_{ij} \in S_{ij}$, $a_{ii} = e_i$ and

(11) $$\varphi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k} = a_{i,jk}\psi_{ijk,jk}(\varphi_{jk,j}(s_j)a_{jk})\ ,$$

for every $i, j, k \in I$. Define a multiplication $*$ on S by:

(12) $$s_i * s_j = \varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j)\ ,\qquad s_i \in S_i,\ s_j \in S_j\ .$$

Then $(S,*)$ is an inflation of a band of monoids.

Conversely, every inflation of a band of monoids can be so constructed.

PROOF. Let conditions of this Theorem hold. As in the proof of Theorem 1. we obtain that $S^2 = G = \cup\{G_i \mid i \in I\}$, that $(S,*)$ is a semigroup and that $(G,*)$ is a band of monoids. Define a mapping $\phi : S \to G$ by:

$$\phi(x) = \varphi_{ii} \qquad \text{if } x \in S_i,\ i \in I\ .$$

For $s_i \in S_i$, $s_j \in S_j$ we have that

$$\begin{aligned}\phi(s_i)\phi(s_j) &= \varphi_{ii}(s_i)\psi_{jj}(s_j) \\ &= \varphi_{ij,i} \circ \varphi_{ii}(s_i)a_{ij}\psi_{ij,j} \circ \psi_{jj}(s_j) \\ &= \varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j) = s_i * s_j \\ &= \varphi_{ij,ij}(s_i * s_j) = \phi(s_i * s_j)\ .\end{aligned}$$

Therefore, ϕ is a retraction of S onto G, so S is an inflation of a band of monoids G.

Conversely, let S be an inflation of a band I of monoids G_i, $i \in I$. In a similar way as in the proof of Theorem 1. we determine semigroups S_i, $i \in I$. If we define mappings φ_{ij} and ψ_{ij} over $\leq_1$ and $\leq_2$, respectively, by

$$\varphi_{ij}(s_j) = s_j e_j e_i\ , \qquad \psi_{ij}(s_j) = e_i e_j s_j\ ,$$

then easily we prove the statements (6)-(12). □

THEOREM 3. *Let I be a band. To each $i \in I$ we associate a semigroup S_i such that $S_i \cap S_j = \varnothing$ if $i \neq j$. Let S_i be an inflation of an unipotent monoid G_i. Let φ_{ij} and ψ_{ij} be transitive systems of homomorphisms over $\leq_1$ and $\leq_2$, respectively. Let (a_{ij}) be an $I \times I$ matrix over $S = \cup\{S_i \mid i \in I\}$ such that $a_{ij} \in S_{ij}$, a_{ii} is the identity of G_i and the condition (4) holds. Define a multiplication $*$ on S by (5). Then $(S, *)$ is an inflation of a band of unipotent monoids.*

Conversely, every inflation of a band of unipotent monoids can be so constructed. □

THEOREM 4. *A semigroup S is an inflation of a band of groups if and only if*

$$ab \in a^2bS \cap Sab^2\ , \tag{13}$$

for all $a, b \in S$.

PROOF. Let S be an inflation of a semigroup T with a retraction φ of S onto T, and let T be a band I of groups G_i, $i \in I$. Let $a, b \in S$, and let $\varphi(a) \in G_i$, $\varphi(b) \in G_j$, for some $i, j \in I$. Since ab, a^2b, $ab^2 \in T$, we then obtain that $ab = \varphi(ab) = \varphi(a)\varphi(b) \in G_{ij}$, $a^2b = \varphi(a^2b) = (\varphi(a))^2\varphi(b) \in G_{ij}$ and $ab^2 = \varphi(ab^2) = \varphi(a)(\varphi(b))^2 \in G_{ij}$. Therefore,

$$ab \in a^2bG_{ij}ab^2 \subseteq a^2bSab^2\ .$$

Conversely, let (13) holds. Then by Theorem 1. [3] it follows that S is an inflation of an union of groups T. Let $a, b \in T$ and let $a\,\mathcal{H}\,e$, $b\,\mathcal{H}\,f$, for some $e, f \in E(S)$, where $\mathcal{H}$ is the Green's relation on T. Then

$$ab = a^2a^{-1}b \in a^2(a^{-1})^2bS = ebS = efbS \subseteq efT\ ,$$

$$ab = ab^{-1}b^2 \in Sa(b^{-1})^2b^2 = Saf = Saef \subseteq Tef\ ,$$

$$ef = a^{-1}af \in a^{-1}a^2fS = afS = ab^{-1}bfS \subseteq ab^{-1}b^2fS = abfS \subseteq abT\ ,$$

$$ef = ebb^{-1} \in Seb^2b^{-1} = Seb = Seaa^{-1}b \subseteq ea^2a^{-1}b = Seab \subseteq Tab\ .$$

Therefore, $ab\,\mathcal{H}\,ef$, so $\mathcal{H}$ is a congruence. Hence, T is a band of groups. □

Rédei's bands of semigroups are the interesting class of semigroups, with very important applications. For one of its see [7,Chapter V]. Here we consider inflations of Rédei's bands of monoids and inflations of Rédei's bands of groups.

THEOREM 5. *Let* I *be a Rédei's band. To each* $i \in I$ *we associate a monoid* S_i *with the identity* e_i *such that* $S_i \cap S_j = \varnothing$ *if* $i \neq j$. *Let* φ_{ij} *and* ψ_{ij} *be systems of homomorphisms over* $\leq_1$ *and* $\leq_2$, *respectively, for which the following properties hold:*

(14) φ_{ij} *and* ψ_{ij} *are the identical automorphisms of* S_i, *for every* $i \in I$;

(15) $$\varphi_{ij} \circ \varphi_{jk}(s_k) = \varphi_{ik}(s_k)\varphi_{ij}(e_j) , \qquad i \leq_1 j \leq_1 k ,$$

(16) $$\psi_{ij} \circ \psi_{jk}(s_k) = \psi_{ij}(e_j)\psi_{ik}(s_k) , \qquad i \leq_2 j \leq_2 k .$$

(17) $$\psi_{ij} \circ \varphi_{jk}(s_k) = \psi_{ik}(s_k)\psi_{ij}(e_j) , \qquad i \leq_2 j \leq_2 k,\ i \leq_2 k ,$$

(18) $$\varphi_{ij} \circ \psi_{jk}(s_k) = \varphi_{ij}(e_j)\varphi_{ik}(s_k) , \qquad i \leq_1 j \leq_1 k,\ i \leq_1 k .$$

Define a multiplication $*$ *on* $S = \cup\{S_i \mid i \in I\}$ *by:*

(19) $$s_i * s_j = \varphi_{ij,i}(s_i)\psi_{ij,j}(s_j) , \qquad s_i \in S_i,\ s_j \in S_j .$$

Then $(S,*)$ *is a Rédei's band of monoids.*

Conversely, every Rédei's band of monoids can be so constructed.

PROOF. Let $s_i \in S_i$, $s_j \in S_j$ and $s_k \in S_k$. Then

$$\begin{aligned}(s_i * s_j) * s_k &= (\varphi_{ij,i}(s_i)\psi_{ij,j}(s_j)) * s_k\\ &= \varphi_{ijk,ij}(\varphi_{ij,i}(s_i)\psi_{ij,j}(s_j))\psi_{ijk,k}(s_k)\\ &= \varphi_{ijk,ij} \circ \varphi_{ij,i}(s_i)\varphi_{ijk,ij} \circ \psi_{ij,j}(s_j)\psi_{ijk,k}(s_k)\\ &= \varphi_{ijk,i}(s_i)\varphi_{ijk,ij}(e_{ij})\varphi_{ijk,ij} \circ \psi_{ij,j}(s_j)\psi_{ijk,k}(s_k)\\ &= \varphi_{ijk,i}(s_i)\varphi_{ijk,ij} \circ \psi_{ij,j}(s_j)\psi_{ijk,k}(s_k) ,\\ s_i * (s_j * s_k) &= s_i * (\varphi_{jk,j}(s_j)\psi_{jk,k}(s_k))\\ &= \varphi_{ijk,i}(s_i)\psi_{ijk,jk}(\varphi_{jk,j}(s_j)\psi_{jk,k}(s_k))\\ &= \varphi_{ijk,i}(s_i)\psi_{ijk,jk} \circ \varphi_{jk,j}(s_j)\psi_{ijk,jk} \circ \psi_{jk,k}(s_k)\\ &= \varphi_{ijk,i}(s_i)\psi_{ijk,jk} \circ \varphi_{jk,j}(s_j)\psi_{ijk,jk}(e_{jk})\psi_{ijk,k}(s_k)\\ &= \varphi_{ijk,i}(s_i)\psi_{ijk,jk} \circ \varphi_{jk,j}(s_j)\psi_{ijk,k}(s_k) .\end{aligned}$$

Now we distinguish the following cases:

(i) if $ij = j = jk$, then $ijk = ij = j$, so

$$\varphi_{ijk,ij} \circ \psi_{ij,j} = \varphi_{jj} \circ \psi_{jj} = \psi_{jj} \circ \varphi_{jj} = \psi_{ijk,jk} \circ \varphi_{jk,j} ;$$

(ii) if $ij = j$, $jk = k$, then $ijk = jk = k$, so

$$\varphi_{ijk,ij} \circ \psi_{ij,j} = \varphi_{kj} \circ \psi_{jj} = \psi_{kk} \circ \varphi_{kj} = \psi_{ijk,jk} \circ \varphi_{jk,j} ;$$

(iii) if $ij = i$, $jk = j$, then $ijk = ij = i$, so

$$\varphi_{ijk,ij} \circ \psi_{ij,j} = \varphi_{ii} \circ \psi_{ij} = \psi_{ij} \circ \varphi_{jj} = \psi_{ijk,jk} \circ \varphi_{jk,j} ;$$

(iv) if $ij = i$, $jk = k$, $ik = i$, i.e. $i \leq_2 j$, $i \leq_2 k$, $k \leq_1 j$, then $ijk = ik = i$, so by (17) it follows that

$$\varphi_{ijk,ij} \circ \psi_{ij,j}(s_j) = \varphi_{ii} \circ \psi_{ij}(s_j) ,$$
$$\psi_{ijk,jk} \circ \varphi_{jk,j}(s_j) = \psi_{ik} \circ \varphi_{kj}(s_j) = \psi_{ij}(s_j)\psi_{ik}(e_k) ,$$

whence

$$(s_i * s_j) * s_k = s_i\psi_{ij}(s_j)\psi_{ik}(s_k) = s_i\psi_{ij}(s)j)\psi_{ik}(e_k)\psi_{ik}(s_k)$$

$$= s_i * (s_j * s_k) \ ;$$

(*v*) if $ij = i$, $jk = k$, $ik = k$, i.e. $i \leq_2 j$, $k \leq_1 j$, $k \leq_1 i$, then $ijk = ik = k$, so by (18) it follows that

$$\varphi_{ijk,ij} \circ \psi_{ij,j}(s_j) = \varphi_{ki} \circ \psi_{ij}(s_j) = \varphi_{ki}(e_i)\varphi_{kj}(s_j) \ ,$$
$$\psi_{ijk,jk} \circ \varphi_{jk,j}(s_j) = \psi_{kk} \circ \varphi_{kj}(s_j) = \varphi_{kj}(s_j) \ ,$$

whence

$$(s_i * s_j) * s_k = \varphi_{ki}(s_i)\varphi_{ki}(e_i)\varphi_{kj}(s_j)s_k = \varphi_{ki}(s_i)\varphi_{kj}(s_j)s_k$$
$$= s_i * (s_j * s_k) \ .$$

Therefore, $(S, *)$ is a semigroup and, clearly, S is a Rédei's band I of monoids S_i, $i \in I$. Let e_i be the identity of S_i, $i \in I$. Define mappings φ_{ij} and ψ_{ij} over $\leq_1$ and $\leq_2$, respectively, by:

$$\varphi_{ij}(s_j) = s_j e_i \ , \qquad \psi_{ij}(s_j) = e_i s_j \ .$$

Then by the proof of Theorem 1. [3] we have that φ_{ij} and ψ_{ij} are homomorphisms and that (14),(15) and (16) hold. In a similar way we prove (17) and (18). Finally,

$$s_i s_j = s_i e_i s_j = s_i e_{ij} s_j = s_i e_{ij} e_{ij} s_j \ , \quad \text{if } ij = i \ ,$$
$$s_i s_j = s_i e_j s_j = s_i e_{ij} s_j = s_i e_{ij} e_{ij} s_j \ , \quad \text{if } ij = j \ ,$$

so, in any cases,

$$s_i s_j = s_i e_{ij} e_{ij} s_j = \varphi_{ij,i}(s_i)\psi_{ij,j}(s_j) \ (= s_i * s_j) \ . \ \square$$

THEOREM 6. *Let I be a Rédei's band. To each $i \in I$ we associate a semigroup S_i which is an inflation of a monoid G_i with the identity e_i, such that $S_i \cap S_j = \varnothing$ if $i \neq j$. Let φ_{ij} and ψ_{ij} be systems of homomorphisms over $\leq_1$ and $\leq_2$, respectively, for which the following properties hold:*

(20) *for every $i \in I$ $\varphi_{ij} = \psi_{ij}$ is a retraction of S_i onto G_i;*

(21) $$\varphi_{ii} \circ \varphi_{ij} = \varphi_{ij} \circ \varphi_{jj} = \varphi_{ij} \ , \qquad i \leq_1 j \ ,$$

(22) $$\psi_{ii} \circ \psi_{ij} = \psi_{ij} \circ \psi_{jj} = \psi_{ij} \ , \qquad i \leq_2 j \ ,$$

(23) $$\varphi_{ij} \circ \varphi_{jk}(s_k) = \varphi_{ik}(s_k)\varphi_{ij}(e_j) \ , \qquad i <_1 j <_1 k \ ,$$

(24) $$\psi_{ij} \circ \psi_{jk}(s_k) = \psi_{ij}(e_j)\psi_{ik}(s_k) \ , \qquad i <_2 j <_2 k \ .$$

(25) $$\psi_{ij} \circ \varphi_{jk}(s_k) = \psi_{ik}(s_k)\psi_{ij}(e_j) \ , \qquad i \leq_2 j \leq_2 k, \ i \leq_2 k \ ,$$

(26) $$\varphi_{ij} \circ \psi_{jk}(s_k) = \varphi_{ij}(e_j)\varphi_{ik}(s_k) \ , \qquad i \leq_1 j \leq_1 k, \ i \leq_1 k \ .$$

Define a multiplication $*$ on $S = \cup\{S_i \mid i \in I\}$ by:

$$s_i * s_j = \varphi_{ij,i}(s_i)\psi_{ij,j}(s_j) \ , \qquad s_i \in S_i, \ s_j \in S_j \ .$$

Then $(S, *)$ is an inflation of a Rédei's band of monoids.

Conversely, every inflation of a Rédei's band of monoids can be so constructed.

PROOF. Let $s_i \in S_i$, $s_j \in S_j$ and $s_k \in S_k$. Then

$$(s_i * s_j) * s_k = \varphi_{ijk,ij} \circ \varphi_{ij,i}(s_i)\varphi_{ijk,ij} \circ \psi_{ij,j}(s_j)\psi_{ijk,k}(s_k)$$
$$= \varphi_{ijk,i}(s_i)\varphi_{ijk,ij}(e_{ij})\varphi_{ijk,ij} \circ \psi_{ij,j}(s_j)\psi_{ijk,k}(s_k)$$
$$= \varphi_{ijk,i}(s_i)\varphi_{ijk,ij} \circ \varphi_{ij,ij} \circ \psi_{ij,j}(s_j)\psi_{ijk,k}(s_k) \ ,$$
$$= \varphi_{ijk,i}(s_i)\varphi_{ijk,ij} \circ \psi_{ij,j}(s_j)\psi_{ijk,k}(s_k) \ ,$$
$$s_i * (s_j * s_k) = \varphi_{ijk,i}(s_i)\psi_{ijk,jk} \circ \varphi_{jk,j}(s_j)\psi_{ijk,k}(s_k)) \ .$$

Now we have the same cases as in the proof of Theorem 5. and in cases $(i), (ii)$ and (iii) we have similar proofs. Let we consider the case (iv). Then

$$\varphi_{ijk,ij} \circ \psi_{ij,j} = \varphi_{ii} \circ \psi_{ij} = \psi_{ij} ,$$
$$\psi_{ijk,jk} \circ \varphi_{jk,j} = \psi_{ik} \circ \varphi_{kj} ,$$

whence

$$\begin{aligned} s_i * (s_j * s_k) &= \varphi_{ii}(s_i)\psi_{ik} \circ \varphi_{kj}(s_j)\psi_{ik}(s_k) \\ &= \varphi_{ii}(s_i)\psi_{ij}(s_j)\psi_{ik}(e_k)\psi_{ik}(s_k) \\ &= \varphi_{ii}(s_i)\psi_{ij}(s_j)\psi_{ik} \circ \psi_{kk}(s_k) \\ &= \varphi_{ii}(s_i)\psi_{ij}(s_j)\psi_{ik}(s_k) = (s_i * s_j) * s_k . \end{aligned}$$

The similar proof we have in the case (v). Thus, $(S, *)$ is a semigroup. Let $G = \cup\{G_i \mid i \in I\}$. By Theorem 5. it follows that $(G, *)$ is a Rédei's band of monoids. In a similar way as in the proof of Theorem 2. we shows that S is an inflation of G.

Conversely, let S be an inflation of a band I of monoids G_i, $i \in I$, with the retraction ϕ. Using methods and definitions from the proof of Theorem 2, we easy prove that conditions (20)-(26) hold. Let $s_i \in S_i$, $s_j \in S_j$. Then

$$\begin{aligned} s_i s_j &= \phi(s_i)\phi(s_j) = s_i e_i e_j s_j \\ &= \begin{cases} s_i e_i e_i e_i e_j s_j = s_i e_i e_{ij} e_{ij} e_j s_j & \text{if } ij = i , \\ s_i e_i e_j e_j e_j s_j = s_i e_i e_{ij} e_{ij} e_j s_j & \text{if } ij = j , \end{cases} \\ &= \varphi_{ij,i}(s_i)\psi_{ij,j}(s_j) \quad (= s_i * s_j). \ \square \end{aligned}$$

REMARK 1. If φ_{ij} and ψ_{ij} be systems from Theorem 6. and we put that

$$a_{ij} = \psi_{ij}(e_i) \quad \text{if } i \leq_2 j , \quad a_{ij} = \varphi_{ji}(e_i) \quad \text{if } j \leq_1 i ,$$

then we can prove that systems φ_{ij} and ψ_{ij} and the matrix (a_{ij}) satisfy the conditions of Theorem 2, so Theorem 6. is a special case of Theorem 2. In this way we obtain, also, that Theorem 5. is a special case of Theorem 1. [2].

REMARK 2. If φ_{ij} and ψ_{ij} be systems from Theorem 6. then in the case $i \leq_1 j$ and $i \leq_2 j$ we obtain that $\varphi_{ij} = \psi_{ij}$. Clearly, that holds also for systems of homomorphisms from Theorem 5.

THEOREM 7. *A semigroup S is an inflation of a Rédei's band of groups if and only if for all $a, b \in S$ one of the following conditions holds:*

$$ab \in a^2 S a^2 \quad \wedge \quad a^2 \in Sb ; \tag{27}$$
$$ab \in b^2 S b^2 \quad \wedge \quad b^2 \in aS . \tag{28}$$

PROOF. Let S be an inflation of a semigroup T with the retraction φ of S onto T and let T be a Rédei's band I of groups G_i, $i \in I$. Let $a, b \in S$ and let $\varphi(a) \in G_i$, $\varphi(b) \in G_j$ for some $i, j \in I$. If $ij = i$, then

$$ab = \varphi(ab) = \varphi(a)\varphi(b) \in G_{ij} = G_i ,$$
$$a^2 = \varphi(a^2) = (\varphi(a))^2 \in G_i ,$$

whence

$$ab \in a^2 G_i a^2 \quad \text{and} \quad a^2 \in abG_i ab ,$$

so (27) holds. If $ij = j$, then in a similar way we prove that

$$ab \in b^2G_jb^2 \quad \text{and} \quad b^2 \in abG_jab \ ,$$

so (28) holds.

Conversely, let for all $a, b \in S$ one of the conditions (27) and (28) holds. Then $a^2 \in a^2Sa^2$, i.e. a^2 is regular for all $a \in S$. By hypothesis and by Theorem 3.1. [4] we have that S is a semilattice Y of semigroups S_α, $\alpha \in Y$, and for every $\alpha \in Y$, S_α is a nil-extension of a left or a right group K_α. Let $a \in S_\alpha$, $b \in S_\beta$ for some $\alpha, \beta \in Y$, and let $a^2 \in G_e$, $b^2 \in G_f$ for some $e, f \in E(S)$. Assume that (27) holds, i.e. that $ab = a^2ua^2$ and $a^2 = vb$ for some $u, v \in S$. Then it is easy to verify that $\alpha\beta = \beta\alpha = \alpha$, so by Lemma 1. [5] it follows that

$$ab = a^2ua^2 = ea^2ua^2e \in eS_\alpha e = G_e \ .$$

and $a^2b = a(ab) \in G_e$, so

$$e \in Sa^2b = Svb^2 \ ,$$

whence $ef = e$. Therefore,

$$ab \in G_{ef} \ . \tag{29}$$

In a similar way we show that by (28) it follows that (29) holds and $ef = f$. Hence, S^2 is a Rédei's band $E(S)$ of groups G_e, $e \in E(S)$. Define a mapping $\varphi : S \to S^2$ by:

$$\varphi(x) = xe \qquad \text{if } x^2 \in G_e, \ e \in E(S) \ .$$

Let $a, b \in S$, $a^2 \in G_e$, $b^2 \in G_f$. Then by (29) it follows that $ab \in G_{ef}$, so by Lemma 1. [6] we obtain that

$$\varphi(ab) = abef = eabf = aebf = \varphi(a)\varphi(b) \ ,$$

if $ab \in G_e$ (for $ef = e$), and

$$\varphi(ab) = abef = efab = eabf = aebf\varphi(a)\varphi(b) \ ,$$

if $ab \in G_f$ (for $ef = f$). Therefore, S is an inflation of a Rédei's band of groups. □

Corollary 1. *A semigroup S is an inflation of a Rédei's band of periodic groups if and only if*

$$ab \in \langle a\rangle^2 \cup \langle b\rangle^2$$

for all $a, b \in S$. □

Corollary 2. *Let I be a Rédei's band. To each $i \in I$ we associate a group S_i such that $S_i \cap S_j = \varnothing$ if $i \neq j$. Let φ_{ij} be a transitive system of homomorphisms over the quasiorder $\leq_3$. Define a multiplication $*$ on $S = \cup\{S_i \mid i \in I\}$ by:*

$$s_i * s_j = \varphi_{ij,i}(s_i)\varphi_{ij,j}(s_j) \ , \quad s_i \in S_i, \ s_j \in S_j \ .$$

Then $(S, *)$ is a Rédei's band of groups.

Conversely, every Rédei's band of groups can be so constructed. □

Corollary 3. *Let I be a Rédei's band. To each $i \in I$ we associate an inflation S_i of a group G_i such that $S_i \cap S_j = \varnothing$ if $i \neq j$. Let φ_{ij} be a system of homomorphisms over the quasiorder $\leq_3$ for which the following properties hold:*

(i) *for every* $i \in I$ φ_{ij} *is a retraction of* S_i *onto* G_i;

(ii) $$\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} , \qquad i \leq_3 j \leq_3 k .$$

Define a multiplication $*$ *on* $S = \cup\{S_i \mid i \in I\}$ *by:*

$$s_i * s_j = \varphi_{ij,i}(s_i)\varphi_{ij,j}(s_j) , \quad s_i \in S_i,\ s_j \in S_j .$$

Then $(S, *)$ *is an inflation of a Rédei's band of groups.*

Conversely, every inflation of a Rédei's band of groups can be so constructed.

PROOF. This follows by Theorem 6, Remark 2. and by the fact that homomorphic image of an idempotent is an idempotent. □

REFERENCES

[1] S.BOGDANOVIĆ, *Semigroups with a system of subsemigroups*, Inst. of Math. Novi Sad, 1985.

[2] S.BOGDANOVIĆ, *Inflation of a union of groups*, Matematički Vesnik, **37** (1985), 351-355.

[3] S.BOGDANOVIĆ AND M.ĆIRIĆ, *Bands of monoids*, Matematički bilten Skopje, **9-10** (XXXV-XXXVI) (1985-1986), 57-61.

[4] S.BOGDANOVIĆ AND M.ĆIRIĆ, *Semigroups of Galbiati-Veronesi* III (*Semilattice of nil-extensions of left and right groups*), Facta Univ. Niš, Ser. Math. Inform. 4 (1989), 1-14.

[5] S.BOGDANOVIĆ AND S.MILIĆ, *A nil-extension of a completely simple semigroup*, Publ. Inst. Math. 36 (50), (1984), 45-50.

[6] W.D.MUNN, *Pseudoinverses in semigroups*, Proc. Camb. Phil. Soc. 57 (1961), 247-250.

[7] M.PETRICH, *Lectures in semigroups*, Akad. Verlag, Berlin, 1977.

[8] B.M.SCHEIN, *Bands of monoids*, Acta Sci. Math. Szeged **36** (1974), 145-154.

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