

SUBDIVISION OF BERNSTEIN-BEZIER OPERATORS USING MULTIAFFINE MAPS

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ABSTRACT. *A generalized subdivision for Bernstein-Bezier one- and two-dimensional triangular operators in the domain of images is obtained. Some applications in geometric modelling of curve segments and triangular patches are given.*

1. Introduction. The technique of multiaffine mapping is applied to the analysis of the difference formulas of the form $B_{n+1} - B_n$, where B_n is n -th Bernstein-Bezier operator

$$(1) \quad B_n : \{P_{ij}\} \rightarrow F_n, \quad P_{ij} \in L, \quad i, j \in \{0, \dots, n\}, \quad i + j = n,$$

which maps an affine space L (of an arbitrary finite dimension), in the space of parametric polynomials of degree not greater than n . A polynomial of order n is any function $F_n : I \rightarrow L$, where I is a finite dimensional affine space, provided that any Descartes coordinate of the point $F_n(u)$ is a polynomial in the coordinates of the point u . Note that this condition does not depend on choosing the coordinate origin in I or L . In the case $\dim(I) = 1$, F_n is a polynomial curve while for $\dim(I) = 2$, it is a polynomial surface. The space I is an affine parametric space for F_n mostly being a parametric polynomial map. Subdivision of the operator B_n is any map of the type (1) so that

$$(2) \quad B_n^J : \{P_{ij}^J\} \rightarrow F_n^J, \quad P_{ij}^J \in L, \quad i + j = n,$$

where F_n^J is a restriction of a polynomial F_n on the affine parametric subspace $J \subset I$. The function $F : I \rightarrow L$ is an affine if it preserves affine combination, i.e. if

$$F((1 - \lambda)u + \lambda v) = (1 - \lambda)F(u) + \lambda F(v), \quad \lambda \in R,$$

The n -variable function is multiaffine or n -affine if it is affine on each of its arguments when other are fixed. It is known that the set of polynomials of degree n is equivalent to the set of n -affine mappings. This equivalence can be applied very fruitfully to studying parametric polynomial curves and surfaces, especially in computer aided geometric modeling applications. That was firstly observed by a French mathematician Paul de Faget de Casteljau [1]. Similar ideas for B -splines go back to Carl de Boor in his works from 1986 and 1987. An exhaustive three-year study by Lyle Ramshaw [4] crowned the theory. According to Ramshaw's words,

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his work was initialized by one-day search for a good way to label some diagrams of splines. His research is performed independently of de Casteljau and de Boor. We shall use two theorems from [5]:

THEOREM 1. *Every polynomial $F : I \rightarrow L$ of degree n is equivalent to the symmetric, n -affine map $f : I^n \rightarrow L$. In particular, given a function of either type, a unique function of the other type exists, so that the identity*

$$(3) \quad f(u, \dots, u) = F(u).$$

is satisfied. Then, f is the multiaffine blossom or polar form of the polynomial F , while F is a diagonal of f .

The operation of finding f for a given F is known as polarization. The reverse operation, defined by (3) is diagonalization.

Polarization can be performed in a few ways (in [6] Ramshaw suggest eight) but the simplest way uses the elementary symmetric functions

$$\sigma_k(u_1, \dots, u_n) = \sum_{j_1 + \dots + j_n = k} u_1^{j_1} * \dots * u_n^{j_n}, \quad j_r \in \{0, 1\},$$

and replace t^k in the monomial representation of the polynomial $t \rightarrow F_n(t)$ by the term $\sigma_k(u_1, \dots, u_n) / \binom{n}{k}$, $k = 0, \dots, n$. But no simple relationship between the coefficients of F_n , represented via monomial basis $\{1, t, t^2, \dots, t^n\}$, and its polar form f is known. On the other hand, representation via Bernstein basis $\{B_{0n}^n(t), \dots, B_{nn}^n(t)\}$ is given by

$$(4) \quad B_{ij}^n(t) = \frac{n!}{i!j!} u^i v^j, \quad i + j = n, \quad t \in I = [a, b],$$

where (u, v) are the barycentric coordinates of the point $t \in [a, b]$, $u = (t-a)/(b-a)$, $v = (b-t)/(b-a)$. It allows a very simple relation between the coefficients P_{ij} of the polynomial

$$(5) \quad F_n(t) = \sum_{i+j=n} P_{ij} B_{ij}^n(t), \quad t \in [a, b],$$

and its polar form.

THEOREM 2. *If F_n is a n -degree polynomial given by (5) and f is its polar form, then*

$$(6) \quad P_{ij} = f(\underbrace{a, \dots, a}_j, \underbrace{b, \dots, b}_i).$$

It is customary to call P_{ij} the Bézier points, or poles.

A characteristic that gives an extraordinary power to the theory of multiaffine maps is its independence of the affine parametric space I . So, for example, in the case when $\dim(I) = 2$, the analogs of (5) and (6) are valid for the operators

$$(7) \quad B_n^\Delta : \{P_{ij}\} \rightarrow F_n^\Delta, \quad P_{ijk} \in L,$$

where F_n^Δ are two-variable polynomial on the triangle domain Δ from R^2 . These are known as the Bernstein-Bézier operators on triangles. Polynomials F_n^Δ are given by

$$(8) \quad F_n^\Delta(p) = \sum_{i+j+k=n} P_{ijk} B_{ijk}^n(p), \quad p \in \Delta,$$

where the basis polynomials

$$(9) \quad B_{ijk}^n(p) = \frac{n!}{i!j!k!} u^i v^j w^k, \quad p \in \Delta$$

are functions of barycentric coordinates (u, v, w) of the point p with respect to the triangular domain Δ . If f^Δ is the polar form of the polynomial F_n^Δ , then the Bézier points P_{ijk} are given by

$$(10) \quad P_{ijk} = f(\underbrace{r, \dots, r}_i, \underbrace{s, \dots, s}_j, \underbrace{t, \dots, t}_k)$$

where r, s and t are vertices of the triangular fragment Δ .

2. Difference formulae and subdivision. Consider two Bernstein-Bézier operators B_P and B_Q of the form (1) where $P = \{P_{ij}\}_{i+j=n}$, $Q = \{Q_{ij}\}_{i+j=n+1}$, so that $t \rightarrow B_P(t)$ and $t \rightarrow B_Q(t)$ are corresponding polynomials of degree n and $n+1$ respectively. The difference $B_Q - B_P$ is also the polynomial of degree $n+1$, for ex.

$$B_R(t) = B_Q(t) - B_P(t), \quad t \in [a, b],$$

or, after polarization

$$b_Q(u_1, \dots, u_{n+1}) = b_P(u_1, \dots, u_n) + b_R(u_1, \dots, u_{n+1}),$$

which, after raising the degree of b_P for one (p. 58 [4]) gives

$$(11) \quad b_Q(u_1, \dots, u_{n+1}) = \frac{1}{n+1} \sum_{k=1}^{n+1} b_P(u_1, \dots, \hat{u}_k, \dots, u_{n+1}) + b_R(u_1, \dots, u_{n+1}),$$

where \hat{u}_k means that variable u_k should be omitted. Substituting $u_1, \dots, u_j = a$, $u_{j+1}, \dots, u_n = b$ in (11) and applying Theorem 2, we get

$$(12) \quad Q_{ij} = \frac{i}{n+1} P_{i-1,j} + \frac{j}{n+1} P_{i,j-1} + R_{ij}, \quad i+j = n+1.$$

It is shown in [2] that, under the conditions $P_{0,n} = Q_{0,n+1}$, $P_{n,0} = Q_{n+1,0}$, the poles R_{ij} in (11) have the representation

$$(13) \quad R_{ij} = \frac{ij}{n^2(n+1)^2} a_{ij},$$

where a_{ij} is an arbitrary set of vectors from the linear space, associated with the affine space L . Similar is valid for polynomials over a triangular domain $\Delta = (r, s, t)$. Namely, from the difference

$$B_Q^\Delta(p) - B_P^\Delta(p) = B_R^\Delta(p), \quad p \in \Delta$$

one can derive

$$(14) \quad b_Q^\Delta(u_1, \dots, u_{n+1}) = \frac{1}{n+1} \sum_{k=1}^{n+1} b_P^\Delta(u_1, \dots, \hat{u}_k, \dots, u_{n+1}) b_R^\Delta(u_1, \dots, u_{n+1}),$$

which differs from (11) in variables u_k being ordered pairs $u_k = (u_k^1, u_k^2)$ rather than numbers. So, keeping the first multiset of i variables on the value r , the second multiset of j variables on s and the third on t , we get

$$(15) \quad Q_{ijk} = \frac{i}{n+1} P_{i-1,j,k} + \frac{j}{n+1} P_{i,j-1,k} + P_{i,j,k-1} + R_{ijk}, \quad i+j+k = n+1.$$

It has been shown in [2] that, under conditions $P_{0,0,n} = Q_{0,0,n+1}$, $P_{0,n,0} = Q_{0,n+1,0}$, $P_{n,0,0} = Q_{n+1,0,0}$ it holds

$$(16) \quad R_{ijk} = \frac{ij + ik + jk}{n^2(n+1)^2} a_{ijk},$$

where $\{a_{ijk}\}$, $i+j+k = n+1$ is an arbitrary set of vectors. Suppose B_P^J is a Bernstein-Bézier polynomial derived by subdivision of the operator B_n^J , given by (2). In the case that $J = [c, d] \subset [a, b]$, the poles of the restriction B_P^J are given by

$$(17) \quad P_{ij}^J = f(\underbrace{c, \dots, c}_j, \underbrace{d, \dots, d}_i).$$

Similarly, if J is a triangle (r_1, s_1, t_1) that lies in the triangle $\Delta = (r, s, t)$,

$$(18) \quad P_{ijk}^J = f(\underbrace{r_1, \dots, r_1}_i, \underbrace{s_1, \dots, s_1}_j, \underbrace{t_1, \dots, t_1}_k).$$

Combining (12), (13), (15) and (16) we get an essential generalization of the subdivision formula for P_Q^J

$$(19) \quad Q_{ij}^J = \frac{i}{n+1} P_{i-1,j}^J + \frac{j}{n+1} P_{i,j-1}^J + \frac{ij}{n^2(n+1)^2} a_{ij}^J,$$

i.e.

$$(20) \quad Q_{ijk}^J = \frac{i}{n+1} P_{i-1,j,k}^J + \frac{j}{n+1} P_{i,j-1,k}^J + \frac{k}{n+1} P_{i,j,k-1}^J + \frac{ij + ik + jk}{n^2(n+1)^2} a_{ijk}^J.$$

3. Applications. Beside the theoretical significance, formulas (19) and (20) have a practical value as well. Namely, if a polynomial curve of order n is given, and if its Bézier points are given, it is possible to calculate new Bézier points P_{ij}^J in a stable and simple way, using (17). These new points, in turn, control the corresponding subsegment of the curve, so, we can change the form of the subsegment which at the time becomes of one degree higher. The iterative application of this algorithm leads to the satisfactory method for modeling a curve segment [2].

The analogous treatment is possible for a triangular fragment $B_P^\Delta(p)$, with the difference that in the case, we have $(n+2)(n+3)/2$ vectors a_{ijk} . Adjusting these vectors separately, or put $a_{ijk} = a$ for all i, j, k allows to change the patch. By (20), we can also change the form of each subpatch B_P^J separately.

As an illustration of our method, we present two examples. In the first one we subdivide the cubic triangular patch with respect to the point from the interior of the domain and then after three subpatches using the algorithm (20). The starting patch is shown at Figure 1 (left) together with the modified patch (right). The second example considers fifth degree patch and its generalized subdivision, see Figure 2.

4. Conclusion. In this paper, the way of deriving some generalized subdivision of the Bernstein-Bézier operator using the technique of multiaffine maps is presented. This technique is particularly efficient when we are working with poles rather than with basic functions. The first one is applicable for one-dimensional Bernstein-Bézier operator, while the second is valid for two-dimensional operator defined on a triangle. Both algorithms describe the same process: rising the degree and modifying the poles of the subdivided restrictions of the operator's representation. Both algorithms include sets of vectors that can be used for adjusting the shape of segment or patch that being graphs of the corresponding operators.

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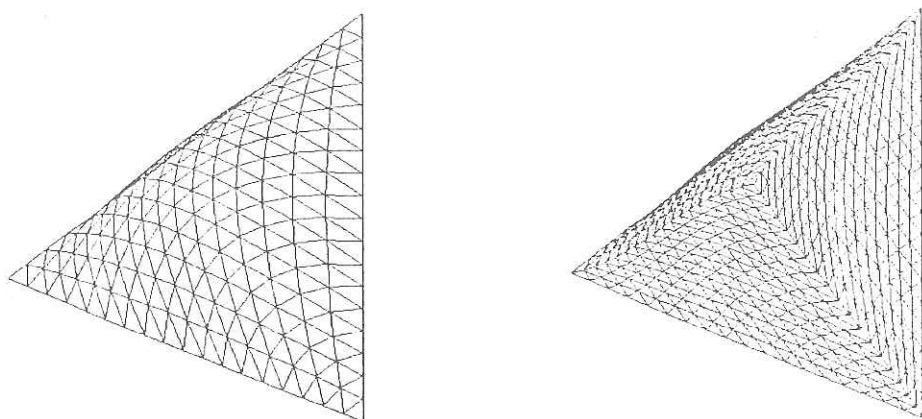


Figure 1. Subdivision of third order Bernstein-Bézier operator. Triangular patch has been subdivided into three subpatches using algorithm (20).

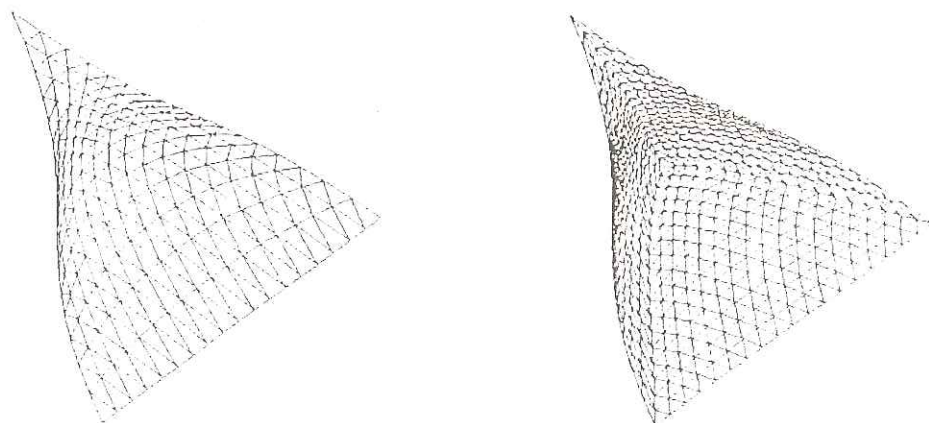


Figure 2. Fifth order patch subdivision and modification of subfragments using algorithm (20).