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SEMIGROUPS IN WHICH THE RADICAL OF EVERY IDEAL IS A SUBSEMIGROUP

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Abstract. *In this paper we consider semigroups* S *in which* $\sqrt{A} = \{x \in S \mid (\exists n \in \mathbb{Z}^+)\ x^n \in A\}$ *is a subsemigroup of* S *for every ideal* A *of* S.

1. Introduction and preliminaries

Throughout paper, by $\mathbb{Z}^+$ we denote the set of all positive integers. If $a, b \in S$, then $a \mid b$ iff $b = xay$ for some $x, y \in S^1$, $a \underset{r}{\mid} b$ iff $ax = b$ for some $x \in S^1$, $a \underset{l}{\mid} b$ iff $xa = b$ for some $x \in S^1$, $a \underset{t}{\mid} b$ iff $a \underset{r}{\mid} b$ and $a \underset{l}{\mid} b$, $a \longrightarrow b$ iff $a \mid b^i$ for some $i \in \mathbb{Z}^+$ and $a \overset{h}{\longrightarrow} b$ iff $a \underset{h}{\mid} b^i$ for some $i \in \mathbb{Z}^+$, where h is r, l or t. A semigroup S is *Archimedean* (*right Archimedean, t-Archimedean, power joined*) iff for all $a, b \in S$, $a \longrightarrow b$ ($a \overset{r}{\longrightarrow} b$, $a \overset{t}{\longrightarrow} b$, $\langle a \rangle \cap \langle b \rangle \neq \emptyset$). By the *radical* of the subset A of a semigroup S we mean the set $\sqrt{A}$ defined by

$$\sqrt{A} = \{x \in S \mid (\exists n \in \mathbb{Z}^+)\ x^n \in A\}.$$

If S is a semigroup with the zero 0, then an element $a \in S$ is a *nilpotent* if there exists $n \in \mathbb{Z}^+$ such that $a^n = 0$, and the set of all nilpotents of S we denote by $Nil(S)$.

T.Tamura and N.Kimura [12] showed that every commutative semigroup is a semilattice of Archimedean semigroups. This well known result has since been generalized by many authors [2-13]. Semigroups which are semilattices of Archimedean semigroups are completely described by M.S.Putcha [7], T.Tamura [10] and by M.Ćirić and S.Bogdanović [4]. M.S.Putcha [7] has proved the following

Theorem P. *A semigroup* S *is a semilattice of Archimedean semigroups if and only if*

$$a \mid b \implies a^2 \longrightarrow b$$

for all $a, b \in S$. □

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These semigroups are, also, completely described by M.Ćirić and S.Bogdanović in [4] by the following

THEOREM ĆB. *The following conditions are equivalent on a semigroup* S*:*

(i) S *is a semilattice of Archimedean semigroups;*

(ii) $(\forall a,b\in S)\ a^2 \longrightarrow ab;$

(iii) *the radical of every ideal of* S *is an ideal of* S. □

Л.Н.Шеврин [14] showed that the equivalence (i) $\iff$ (iii) of Theorem ĆB. holds if S is *completely* π*-regular* $((\forall a\in S)(\exists x\in S)(\exists n\in \mathbf{Z}^+)\ a^n = a^nxa^n,\ a^nx = xa^n)$.

In this paper we characterize semigroups in which the radical of every ideal (right ideal, bi-ideal, subsemigroup) is a subsemigroup (or ideal or bi-ideal or right ideal).

For undefined notions and notations we refer to [1].

2. Main results

THEOREM 1. *The following conditions on a semigroup* S *are equivalent:*

(i) *the radical of every ideal of* S *is a subsemigroup of* S*;*

(ii) *in every homomorphic image with zero of* S *the set of all nilpotent elements is a subsemigroup;*

(iii) $(\forall a,b\in S)(\forall k,l\in \mathbf{Z}^+)\ a^k \longrightarrow ab \quad \vee \quad b^l \longrightarrow ab.$

PROOF. (i) $\Longrightarrow$ (iii). Let $a,b\in S$, $k,l\in \mathbf{Z}^+$. Since $A=S\{a^k,b^l\}S$ is an ideal of S and $a,b\in\sqrt{A}$ we then have by hypothesis that $ab\in\sqrt{A}$. Hence, there exists $n\in\mathbf{Z}^+$ such that $(ab)^n\in S\{a^k,b^l\}S$. Thus $a^k\longrightarrow ab$ or $b^l\longrightarrow ab$.

(iii) $\Longrightarrow$ (ii). Let T be a semigroup with the zero element and let T be a homomorphic image of S. Then the condition (iii) holds in T. For every $a,b\in Nil(T)$ there exist $m,l\in\mathbf{Z}^+$ such that $a^k=b^l=0$, and thus $(ab)^n\in T\{0,0\}T=\{0\}$, for some $n\in\mathbf{Z}^+$. Therefore, $Nil(T)$ is a subsemigroup of T.

(ii) $\Longrightarrow$ (i). Let A be an ideal of S. Let φ be a homomorphism of S onto S/A. Let $a,b\in\sqrt{A}$. Since $\varphi(a),\varphi(b)\in Nil(S/A)$ we then have that $\varphi(a)\varphi(b)\in Nil(S/A)$, i.e. $\varphi(ab)\in Nil(S/A)$ and thus $(ab)^n\in A$ for some $n\in\mathbf{Z}^+$. Hence $ab\in\sqrt{A}$, i.e. $\sqrt{A}$ is a subsemigroup of S. □

In a similar way as in the previous theorem it can be proved the following

THEOREM 2. $\sqrt{R}$ *is a subsemigroup of* S*, for every right ideal* R *of* S *if and only if*

$$(\forall a,b\in S)(\forall k,l\in\mathbf{Z}^+)\ a^k \xrightarrow{r} ab \quad \vee \quad b^l \xrightarrow{r} ab. \ \square$$

THEOREM 3. *The following conditions on a semigroup* S *are equivalent:*

(i) S *is a semilattice of Archimedean semigroups;*

(ii) $(\forall a,b\in S)\ a \mathrel{\underset{r}{|}} b \implies a^2 \longrightarrow b;$

(*iii*) *$\sqrt{SaS}$ is an ideal of S, for all $a \in S$;*

(*iv*) *in every homomorphic image with zero of S the set of all nilpotent elements is an ideal.*

PROOF. By Theorems **P.** and **ĆB.** □

THEOREM 4. *The radical of every right ideal of a semigroup S is a bi-ideal of S if and only if*

$$(1) \qquad (\forall a, b, c \in S)(\forall k, l \in \mathbf{Z}^+)\ a^k \xrightarrow{r} abc \quad \vee \quad c^l \xrightarrow{r} abc\,.$$

PROOF. Let $a, b, c \in S$ and let $k, l \in \mathbf{Z}^+$. Assume that $R = \{a^k, c^l\}S$. Since $a, c \in \sqrt{R}$ and $\sqrt{R}$ is a bi-ideal of S we then have that $abc \in \sqrt{R}S\sqrt{R} \subseteq \sqrt{R}$, i.e. there exists $n \in \mathbf{Z}^+$ such that $(abc)^n \in R = \{a^k, c^l\}S$. Thus $a^k \xrightarrow{r} abc$ or $c^l \xrightarrow{r} abc$.

Conversely, let R be a right ideal of S. For $a, c \in \sqrt{R}$ there exist $k, l \in \mathbf{Z}^+$ such that $a^k, c^l \in R$. Now by (1) we have that

$$\{a^k, c^l\}S \subseteq RS \subseteq R$$

for some $n \in \mathbf{Z}^+$. Hence, $abc \in \sqrt{R}$. Therefore, $\sqrt{R}$ is a bi-ideal of S. □

THEOREM 5. *The following conditions on a semigroup S are equivalent:*

(*i*) *S is a semilattice of right Archimedean semigroups;*

(*ii*) *$(\forall a, b \in S)(\forall k \in \mathbf{Z}^+)\ b^k \xrightarrow{r} ab$;*

(*iii*) *the radical of every right ideal of S is a left ideal of S.*

PROOF. (*i*) $\Longrightarrow$ (*ii*). Let S be a semilattice Y of right Archimedean semigroups S_α, $\alpha \in Y$. Then for $a \in S_\alpha$, $b \in S_\beta$ we have that $ab, b^k a \in S_{\alpha\beta}$, for all $k \in \mathbf{Z}^+$, and there exists $n \in \mathbf{Z}^+$ such that

$$(ab)^n \in b^k a S_{\alpha\beta} \subseteq b^k S\,.$$

Thus $b^k \xrightarrow{r} ab$.

(*ii*) $\Longrightarrow$ (*i*). This implication follows by Proposition 1.1.[2].

(*ii*) $\Longrightarrow$ (*iii*). Let R be a right ideal of S. Assume that $a \in S$, $b \in \sqrt{R}$. Then $b^k \in R$, for some $k \in \mathbf{Z}^+$, and we have that

$$(ab)^n \in b^k S \subseteq RS \subseteq R\,,$$

for some $n \in \mathbf{Z}^+$. Thus $ab \in \sqrt{R}$, i.e. $\sqrt{R}$ is a left ideal of S.

(*iii*) $\Longrightarrow$ (*i*). Let $a, b \in S$, $R = bS$. Then $b \in \sqrt{R}$. Since $\sqrt{R}$ is a left ideal of S we then have that $ab \in \sqrt{R}$, i.e. there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in R = bS$, whence by Proposition 1.1.[2] we have that the condition (*i*) holds. □

THEOREM 6. *The following conditions on a semigroup S are equivalent:*

(*i*) $(\forall a, b \in S)\ a \mid_r b \implies a^2 \xrightarrow{r} b$;

(*ii*) $(\forall a, b \in S)(\forall k \in \mathbf{Z}^+)\ a^k \xrightarrow{r} ab$;

(*iii*) $(\forall a, b \in S)\ a^2 \xrightarrow{r} ab$;

(*iv*) $\sqrt{aS}$ *is a right ideal of* S, *for every* $a \in S$;
(*v*) $\sqrt{R}$ *is a right ideal of* S, *for every right ideal* R *of* S.

PROOF. (*i*) $\Longrightarrow$ (*iii*). Since $ab \in aS$ for every $a, b \in S$, we then have that $(ab)^n \in a^2S$. Thus $a^2 \xrightarrow{r} ab$.

(*iii*) $\Longrightarrow$ (*ii*). By induction.

(*ii*) $\Longrightarrow$ (*i*). Let $b = au$ for some $u \in S$. Then there exists $n \in \mathbf{Z}^+$ such that $b^n = (au)^n \in a^2S$. Thus $a^2 \xrightarrow{r} b$.

(*ii*) $\Longrightarrow$ (*iv*). Let $x \in \sqrt{aS}$ and let $b \in S$. Then $x^k \in aS$ for some $k \in \mathbf{Z}^+$. Since

$$(xb)^n \in x^kS \subseteq aSS \subseteq aS \text{ , for some } n \in \mathbf{Z}^+$$

we then have that $xb \in \sqrt{aS}$. Thus $\sqrt{aS}$ is a right ideal of S.

(*iv*) $\Longrightarrow$ (*iii*). Let $a, b \in S$. Then $a \in \sqrt{a^2S}$. Since $\sqrt{a^2S}$ is a right ideal of S, then $ab \in \sqrt{a^2S}$, and therefore (*iii*) holds.

(*v*) $\Longrightarrow$ (*iv*). Since aS is a right ideal of S, by (*v*) we then have that $\sqrt{aS}$ is also right ideal of S.

(*ii*) $\Longrightarrow$ (*v*). Let R be a right ideal of S. Let $a \in \sqrt{R}$, $b \in S$. Then $a^k \in R$ for some $k \in \mathbf{Z}^+$. Now,

$$(ab)^n \in a^kS \subseteq RS \subseteq R \quad \text{for some } n \in \mathbf{Z}^+$$

and thus $ab \in \sqrt{R}$, i.e. $\sqrt{R}$ is a right ideal of S. □

THEOREM 7. *The following conditions on a semigroup* S *are equivalent:*
(*i*) S *is a normal band of t-Archimedean semigroups;*
(*ii*) $(\forall a, b, c \in S)\ ac \xrightarrow{t} abc$;
(*iii*) *for every* $a, b, c \in S$,

$$a \mathop{|}_{r} c \wedge b \mathop{|}_{l} c \implies ab \xrightarrow{t} c \, .$$

PROOF. (*i*) $\Longleftrightarrow$ (*ii*). This equivalence is from [3].

(*ii*) $\Longrightarrow$ (*iii*). Let $a \mathop{|}_{r} c$, $b \mathop{|}_{l} c$. then there exist $u, v \in S$ such that $c = au = vb$, whence $c^2 = auvb$. Now, there exists $n \in \mathbf{Z}^+$ such that $c^{2n} = (auvb)^n \in abSab$, i.e. $ab \xrightarrow{t} c$.

(*iii*) $\Longrightarrow$ (*ii*). It is clear that $a \mathop{|}_{r} abc$, $c \mathop{|}_{l} abc$, for every $a, b, c \in S$. By (*iii*) there exists $n \in \mathbf{Z}^+$ such that $(abc)^n \in acS \cap Sac$. Hence, (*ii*) holds. □

THEOREM 8. *The following conditions on a semigroup* S *are equivalent:*
(*i*) $(\forall a, b \in S)\ a^2b \xrightarrow{r} ab$;
(*ii*) *for all* $a, b, c \in S$,

$$a \mathop{|}_{r} c \wedge b \mathop{|}_{r} c \implies ab \xrightarrow{r} c \, .$$

PROOF. (*i*) $\Longrightarrow$ (*ii*). Let $c = au = bv$ for some $u, v \in S$, whence $c^2 = (au)^2$. Now, there exists $i \in \mathbf{Z}^+$ such that

$$c^{2i} = (a(uau))^i \in a^2uauS \subseteq a^2uS = a(au)S = a(bv)S \subseteq abS \, .$$

Thus $ab \xrightarrow{r} c$.

$(ii) \implies (i)$. It is clear that $a \underset{r}{|} ab$, $ab \underset{r}{|} ab$, for all $a, b \in S$, and by (ii) we have that $a(ab) = a^2b \xrightarrow{r} ab$. □

THEOREM 9. *The following conditions on a semigroup S are equivalent:*

(i) S is a semilattice of t-Archimedean semigroups;

(ii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in bSa$;

(iii) the radical of every bi-ideal of S is an ideal of S.

PROOF. $(i) \iff (ii)$. This equivalence is from [2].

$(ii) \implies (iii)$. Let A be a bi-ideal of S and let $a \in \sqrt{A}$, $b \in S$. Then $a^k \in A$ for some $k \in \mathbf{Z}^+$, whence

$$(ab)^m, (ba)^n \in a^k bSba^k \subseteq ASA \subseteq A\ ,$$

for some $m, n \in \mathbf{Z}^+$. Thus $ab, ba \in \sqrt{A}$, i.e. $\sqrt{A}$ is an ideal of S.

$(iii) \implies (ii)$. Let $a, b \in S$. Assume that $A = aSa$, $B = bSb$. It is clear that $a \in \sqrt{a}$, $b \in \sqrt{B}$. Since $\sqrt{A}$ and $\sqrt{B}$ are ideals of S, we then have that $ab \in \sqrt{A} \cap \sqrt{B}$, i.e. there exist $m, n \in \mathbf{Z}^+$ such that $(ab)^m \in aSa$, $(ab)^n \in bSb$, whence $(ab)^{m+n} \in bSbaSa \subseteq bSa$. □

3. More on bi-ideals and radicals

The Theorems 10.-22. can be proved as some of the previous theorems and the proof of any of its will be omitted.

THEOREM 10. *The radical of every ideal of a semigroup S is a bi-ideal of S if and only if*

$$(\forall a, b, c \in S)(\forall k, l \in \mathbf{Z}^+)\ a^k \longrightarrow abc \quad \vee \quad c^l \longrightarrow abc. \ \square$$

THEOREM 11. *The radical of every subsemigroup of a semigroup S is a bi-ideal of S if and only if*

$$(\forall a, b, c \in S)(\forall k, l \in \mathbf{Z}^+)(\exists n \in \mathbf{Z}^+)\ (abc)^n \in \langle a^k, c^l \rangle\ . \ \square$$

THEOREM 12. *The radical of every bi-ideal of a semigroup S is a bi-ideal of S if and only if*

$$(\forall a, b, c \in S)(\forall k, l \in \mathbf{Z}^+)(\exists n \in \mathbf{Z}^+)\ (abc)^n \in \{a^k, c^l\}S\{a^k, c^l\}. \ \square$$

THEOREM 13. *The radical of every subsemigroup of a semigroup S is a subsemigroup of S if and only if*

$$(\forall a, b \in S)(\forall k, l \in \mathbf{Z}^+)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in \langle a^k, b^l \rangle\ . \ \square$$

THEOREM 14. *The radical of every bi-ideal of a semigroup S is a subsemigroup of S if and only if*

$$(\forall a, b \in S)(\forall k, l \in \mathbf{Z}^+)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in \{a^k, b^l\}S\{a^k, b^l\}. \ \square$$

THEOREM 15. *The radical of every subsemigroup of a semigroup S is a left ideal of S if and only if S is a right zero band of power joined semigroups.* □

THEOREM 16. *The radical of every subsemigroup of a semigroup S is an ideal of S if and only if S is power joined.* □

THEOREM 17. *The following conditions on a semigroup S are equivalent:*

(i) $(\forall a,b,c,d \in S)\ b^2 \longrightarrow abcd \quad \vee \quad c^2 \longrightarrow abcd$;

(ii) *for all* $a,b,c \in S$,

$$ab \mid c \implies a^2 \longrightarrow c \quad \vee \quad b^2 \longrightarrow c. \ \square$$

THEOREM 18. *The following conditions on a semigroup S are equivalent:*

(i) $(\forall a,b,c \in S)\ a^2 \xrightarrow{r} abc \quad \vee \quad b^2 \xrightarrow{r} abc$;

(ii) *for all* $a,b,c \in S$,

$$ab \mid_r c \implies a^2 \xrightarrow{r} c \quad \vee \quad b^2 \xrightarrow{r} c. \ \square$$

THEOREM 19. *The following conditions on a semigroup S are equivalent:*

(i) $(\forall a,b \in S)\ a^2 \xrightarrow{t} aba$;

(ii) *for all* $a,b \in S$,

$$a \mid_t b \implies a^2 \xrightarrow{t} b. \ \square$$

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