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COMPLETELY REGULAR AND ORTHODOX CONGRUENCES ON REGULAR SEMIGROUPS

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Abstract. *Let* S *be a regular semigroup and let* $E(S)$ *be the set of all idempotents of* S. *Let* $ConS$ *be the congruence lattice of* S *and let* T, K, U *and* V *be equivalences on* $ConS$ *defined by* $\rho T\xi \iff \mathrm{tr}\rho = \mathrm{tr}\xi$, $\rho K\xi \iff \ker\rho = \ker\xi$, $\rho U\xi \iff \rho\cap \leq = \xi\cap \leq$ *and* $V = U \cap K$, *where* $\mathrm{tr}\rho = \rho|_{E(S)}$, $\ker\rho = E(S)\rho$ *and* $\leq$ *is the natural partial order on* $E(S)$. *It is known that* T, U *and* V *are complete congruences on* $ConS$ *and* T-, K-, U- *and* V-*classes are intervals* $[\rho_T, \rho^T]$, $[\rho_K, \rho^K]$, $[\rho_U, \rho^U]$ *and* $[\rho_V, \rho^V]$, *respectively* ([11], [9], [8]). *It turns out that the union of* U-*classes for which* ρ^U *is a semilattice congruence is the lattice* $CRConS$ *of all completely regular congruences on* S *and the union of* V-*classes for which* ρ^V *is an inverse congruence is the lattice* $OConS$ *of all orthodox congruences on* S. *Also, several complete epimorphisms of the form* $\rho \to \rho^U$ *and* $\rho \to \rho^V$ *are obtained.*

1. Preliminaries

In the following we shall use the terminology and notations from [4] and [10]. Throughout the paper, S stands for a regular semigroup. If $\rho \in ConS$ and α is an equivalence on S/ρ, then the equivalence $\overline{\alpha}$ on S defined by $a\,\overline{\alpha}b \iff (a\rho)\,\alpha\,(b\rho)$, $(a, b \in S)$. If α is any relation on S then α^* denotes the congruence on S generated by α. If α is an equivalence on S then α° denotes the greatest congruence on S contained in α and if $T \subseteq S$ then $T\alpha$ denotes the union of α-classes of all elements of T.

Let $\rho \in ConS$. If $\mathfrak{C}$ is a class of semigroups and if $S/\rho \in \mathfrak{C}$, then we say that ρ is a $\mathfrak{C}$-congruence. In the paper, σ, η, α, ν, $\mathcal{Y}$, o denote the least group, semilattice, rectangular band, Clifford, inverse and orthodox congruence, respectively.

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LEMMA 1. *Let* $\mathfrak{C}$ *be a class of semigroups closed for homomorphic images and let* $\gamma = \gamma_S$ *be the least* $\mathfrak{C}$*-congruence on* S. *Then*

(1) ρ *is a* $\mathfrak{C}$*-congruence on* S *if and only if* $\rho \supseteq \gamma$;

(2) $\gamma_{S/\rho} = (\rho \vee \gamma)/\rho$ *for any* $\rho \in ConS$.

For $a \in S$, let $V(a)$ denote the set of all inverses of a in S. Let $\mathcal{U}$ and $\mathcal{V}$ be equivalences on S defined by $a\,\mathcal{U}\,b \iff V(a)\mathcal{H} = V(b)\mathcal{H}$ and $a\,\mathcal{V}\,b \iff V(a) = V(b)$.

RESULT 1. [8] *If* $\rho, \xi \in ConS$, *then*

(1) $\rho T \xi \iff \overline{\mathcal{H}_{S/\rho}} = \overline{\mathcal{H}_{S/\xi}}$ *and* $\rho^T = \overline{\mathcal{H}_{S/\rho}}^{\circ}$, $\rho_T = (\mathrm{tr}\rho)^*$;

(1) $\rho U \xi \iff \overline{\mathcal{U}_{S/\rho}} = \overline{\mathcal{U}_{S/\xi}}$ *and* $\rho^U = \overline{\mathcal{U}_{S/\rho}}^{\circ}$, $\rho_U = (\rho \cap \leq)^*$;

(1) $\rho V \xi \iff \overline{\mathcal{V}_{S/\rho}} = \overline{\mathcal{V}_{S/\xi}}$ *and* $\rho^V = \overline{\mathcal{V}_{S/\rho}}^{\circ}$.

As a consequence we have $\varepsilon^T = \mathcal{H}^\circ$, $\varepsilon^U = \mathcal{U}^\circ$, $\omega_U = (\leq)^*$ and $\varepsilon^V = \mathcal{V}^\circ$, where ε is the equality and ω is the universal relation on S.

RESULT 2. [7] *Let* L *be a complete lattice and let* C *be a complete congruence on* L. *Then for any* $x \in L$ *the* C*-class* xC *is the interval* $[x_C, x^C]$ *of* L. *For any* $A \subseteq L$,

$$(\bigvee_{x \in A} x)_C = \bigvee_{x \in A} x_C\,, \qquad (\bigwedge_{x \in A} x)^C = \bigwedge_{x \in A} x^C\,.$$

2. Completely regular congruences

In this section we establish certain characterizations of completely regular congruences.

LEMMA 2. *For a regular semigroup* S, *the following conditions are equivalent:*

(*i*) S *is completely regular;*

(*ii*) $\eta = \mathcal{U}$;

(*iii*) $\eta \subseteq \mathcal{U}$.

LEMMA 3. *Let* $\rho, \xi \in CRConS$. *Then* $\rho U \xi$ *if and only if* $\rho \vee \eta = \xi \vee \eta$.

THEOREM 1. *For* $\rho \in ConS$, *the following conditions are equivalent:*

(*i*) ρ *is a completely regular congruence;*

(*ii*) ρ^U *is a semilattice congruence;*

(*iii*) $\rho^U = \rho \vee \eta$;

(*iv*) $\rho U (\rho \vee \eta)$.

REMARK 1. From Lemma 3. and by the implication $(i) \implies (iii)$ of Theorem 1. we get Proposition 8.1. of [8].

Let $SConS$ denote the lattice of all semilattice congruences and let $\mu = \varepsilon^T$ be the greatest idempotent separating congruence on S.

COROLLARY 1. *The following conditions hold:*

(1) $CRConS = [\eta_U, \omega] = [\eta, \omega]U$ *and* $\eta U = [\eta_U, \eta]$;

(2) $SConS = [\eta, \omega] = \{\rho \in CRConS \mid \rho^U = \rho\}$;

(3) $\mu \subseteq \mathcal{U}^\circ \subseteq \eta$.

COROLLARY 2. *For* S *the following conditions are equivalent:*

(*i*) S *is completely regular;*

(*ii*) $(\forall \rho \in ConS)\ \rho^U = \rho \vee \eta$;

(*iii*) $\eta = \mathcal{U}^\circ$;

(*iv*) $\varepsilon U \eta$.

The following result is an analogue of Theorem 1. [1].

THEOREM 2. *For* $\rho \in ConS$, *the following conditions are equivalent:*

(*i*) ρ *is a completely simple congruence;*

(*ii*) ρ^T *is a rectangular band congruence;*

(*iii*) $\rho^T = \rho \vee \alpha$;

(*iv*) $\rho T(\rho \vee \alpha)$;

(*v*) $\rho U \omega$.

Let $CSConS$ ($RBConS$ denote the lattice of all completely simple (rectangular band) congruences on S. Then we have

COROLLARY 3. *The following conditions hold:*

(1) $CSConS = [\alpha_T, \omega] = [\alpha, \omega]T$ *and* $\alpha T = [\alpha_T, \alpha]$;

(2) $RBConS = [\alpha, \omega] = \{\rho \in CSConS \mid \rho^T = \rho\}$;

(3) $\alpha_T = (\mathrm{tr}\alpha)^* = (\leq)^*$.

COROLLARY 2. *For* S, *the following conditions are equivalent:*

(*i*) S *is a completely simple;*

(*ii*) $(\forall \rho \in ConS)\ \rho^T = \rho \vee \alpha$;

(*iii*) $\mu = \alpha$;

(*iv*) $\mathcal{U}^\circ = \omega$;

(*v*) $\varepsilon U \omega$.

3. Orthodox congruences

Now we describe orthodox congruences on S in terms of $\mathcal{K}$ and V.

RESULT 3. [3] *If* S *is any orthodox semigroup, then* $\mathcal{V} = \mathcal{Y}$.

LEMMA 4. *Let* $\rho, \xi \in OConS$. *Then* $\rho V \xi$ *if and only if* $\rho \vee \mathcal{Y} = \xi \vee \mathcal{Y}$.

THEOREM 3. *For* $\rho \in ConS$, *the following conditions are equivalent:*

(*i*) ρ *is an orthodox congruence;*

(*ii*) ρ^V *is an inverse congruence;*

(*iii*) $\rho^V = \rho \vee \mathcal{Y}$;

(*iv*) $\rho V(\rho \vee \mathcal{Y})$;

(*v*) $\rho K(\rho \vee \mathcal{Y})$;
(*vi*) ρ^K *is an inverse congruence;*
(*vii*) $\ker\rho$ *is a subsemigroup of* S.

REMARK 2. From Lemma 4. and by the implication $(i) \implies (iii)$ of Theorem 3. we get Proposition 8.5. of [8], and from $(i) \implies (v)$ we get $oK\mathcal{Y}$ [5] and Lemma 2.1. of [2].

Let $IConS$ denote the set of all inverse congruences on S.

COROLLARY 5. *The following conditions hold:*
(1) $OConS = [o, \omega] = [\mathcal{Y}, \omega]V$ *and* $\mathcal{Y}V = [o, \mathcal{Y}]$;
(2) $IConS = [\mathcal{Y}, \omega] = \{\rho \in OConS \mid \rho^V = \rho\}$;
(3) $\mathcal{V}^\circ \subseteq \mathcal{Y}$.

COROLLARY 6. *For* S *the following conditions are equivalent:*
(*i*) S *is orthodox;*
(*ii*) $(\forall \rho \in ConS)\ \rho^V = \rho \vee \mathcal{Y}$;
(*iii*) $\mathcal{Y} \subseteq \mathcal{V}$;
(*iv*) $\mathcal{Y} = \mathcal{V}^\circ$;
(*v*) $\ker\mathcal{Y} = E(S)$;
(*vi*) $\varepsilon V \mathcal{Y}$.

In the following we describe orthodox completely simple and orthodox completely regular congruences.

THEOREM 4. *For* $\rho \in ConS$, *the following conditions are equivalent:*
(*i*) ρ *is a rectangular group congruence;*
(*ii*) ρ^V *is a group congruence;*
(*iii*) $\rho^V = \rho \vee \sigma$;
(*iv*) $\rho V(\rho \vee \sigma)$.

Let $RGConS$ ($GConS$) denote the lattice of all rectangular group (group) congruences on S. Then we have

COROLLARY 7. *The following conditions hold:*
(1) $RGConS = [\sigma_V, \omega] = [\sigma, \omega]V$ *and* $\sigma V = [\sigma_V, \sigma]$;
(2) $GConS = [\sigma, \omega] = \{\rho \in RGConS \mid \rho^V = \rho\}$.

COROLLARY 8. *Let* $\rho, \xi \in RGConS$. *Then* $\rho V \xi$ *if and only if* $\rho \vee \sigma = \xi \vee \sigma$.

COROLLARY 9. *For* S *the following conditions are equivalent:*
(*i*) S *is a rectangular group;*
(*ii*) $(\forall \rho \in ConS)\ \rho^V = \rho \vee \sigma$;
(*iii*) $\mathcal{V}^\circ = \sigma$;
(*iv*) $\varepsilon V \sigma$.

THEOREM 5. *For* $\rho \in ConS$, *the following conditions are equivalent:*

(*i*) ρ *is an orthogroup congruence;*

(*ii*) ρ^V *is a Clifford congruence;*

(*iii*) $\rho^V = \rho \vee \nu$;

(*iv*) $\rho V(\rho \vee \nu)$.

Let $OGConS$ ($SGConS$) denote the lattice of all orthogroup (Clifford) congruences. Then we have

COROLLARY 10. *The following conditions hold:*

(1) $OGConS = [\nu_V, \omega] = [\nu, \omega]V$ *and* $\nu V = [\nu_V, \nu]$;

(2) $SGConS = [\nu, \omega] = \{\rho \in OGConS \mid \rho^V = \rho\}$.

COROLLARY 11. *Let* $\rho, \xi \in OGConS$. *Then* $\rho V \xi$ *if and only if* $\rho \vee \nu = \xi \vee \nu$.

COROLLARY 12. *For* S *the following conditions are equivalent:*

(*i*) S *is an orthogroup;*

ii) $(\forall \rho \in ConS)\ \rho^V = \rho \vee \nu$;

(*iii*) $\mathcal{V}^\circ = \nu$;

(*iv*) $\varepsilon V \nu$.

4. Some complete epimorphisms

Using the results of Theorems 1-5. and the Result 2. we get the following

THEOREM 6. *Let* S *be a regular semigroup. The mappings*

φ_1: $CRConS \to SConS$ *defined by* $\varphi_1(\rho) = \rho \vee \eta$,

φ_2: $CSConS \to RBConS$ *defined by* $\varphi_2(\rho) = \rho \vee \alpha$,

φ_3: $OConS \to IConS$ *defined by* $\varphi_3(\rho) = \rho \vee \mathcal{Y}$,

φ_4: $RGConS \to GConS$ *defined by* $\varphi_4(\rho) = \rho \vee \sigma$,

φ_5: $OGConS \to SGConS$ *defined by* $\varphi_5(\rho) = \rho \vee \nu$,

are complete epimorphisms. The classes of the complete congruence $\overline{\varphi}$ *induced by the epimorphism* φ *are* U*-classes, if* $\varphi = \varphi_1$, T*-classes, if* $\varphi = \varphi_2$ *and* V*-classes, if* $\varphi = \varphi_i$, $i = 3, 4, 5$.

COROLLARY 13. *The following conditions hold:*

(1) $(\bigcap_{\rho \in F} \rho) \vee \eta = \bigcap_{\rho \in F} (\rho \vee \eta)$, $(F \subseteq CRConS)$;

(2) $(\bigcap_{\rho \in F} \rho) \vee \alpha = \bigcap_{\rho \in F} (\rho \vee \alpha)$, $(F \subseteq CSConS)$;

(3) $(\bigcap_{\rho \in F} \rho) \vee \mathcal{Y} = \bigcap_{\rho \in F} (\rho \vee \mathcal{Y})$, $(F \subseteq OConS)$;

(4) $(\bigcap_{\rho \in F} \rho) \vee \sigma = \bigcap_{\rho \in F} (\rho \vee \sigma)$, $(F \subseteq RGConS)$;

(5) $(\bigcap_{\rho \in F} \rho) \vee \nu = \bigcap_{\rho \in F} (\rho \vee \nu)$, $(F \subseteq OGConS)$.

REMARK 3. From (*i*) of this corollary we get Theorem 4.7. of [6], and from (*iii*) we get Theorem 2.4. of [2].

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