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THE CARLESON MEASURE, THE NEVANLINNA-AHLFORS-SHIMIZU CHARACTERISTIC FUNCTION AND APPLICATIONS

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Abstract. *In this paper, I have gathered together some results I have proved for the Carleson measures in the unit disc $|z|<1$ in terms of a new characteristic function. In the case of meromorphic functions, the introduced characteristic function becomes the Nevanlinna characteristic function in the form of Ahlfors-Shimizu. The new characteristic function is used to obtain the necessary and sufficient conditions for a meromorphic function to belong to the class BC of functions with the bounded Nevanlinna characteristic, to the class UBC of functions with the uniformly bounded Nevanlinna characteristic, and for a holomorphic function to belong to the Hardy spaces H^p, $0<p<\infty$, to the hyperbolic Hardy classes H_h^p, $0<p<\infty$, to the classes BMO and $BMOA$.*

1. Preliminaries. For a measurable function $u(z)\geq 0$ defined in the unit disc $D:|z|<1$ on the complex z-plane, we introduce the characteristic function $P(r,u)$ in the form

$$P(r,u)=\int_0^r \frac{S(t,u)}{t}\,dt,\quad 0<r<1,$$

where

$$S(t,u)=\frac{1}{\pi}\iint\limits_{|z|<t}[u(z)]^2\,dxdy,\quad z=x+iy,\quad 0<t<1,$$

and put $P(1,u)=\lim\limits_{r\to 1}P(r,u)$.

If $f(z)$ is a meromorphic function in D and

$$f_p^{\#}(z)=\frac{1}{2}p\,|f'(z)|^{\frac{p}{2}-1}\,|f'(z)|(1+|f(z)|^p)^{-1},\quad 0<p<\infty,$$

then $P(r,f_p^{\#}(z))=\frac{1}{2}p\,T_p(r,f)$, where $T_p(r,f)$ is the characteristic for the meromorphic function $f(z)$ introduced by S. Yamashita [5]; for $p=2$ we get $T_2(r,f)=T(r,f)$, the Nevanlinna characteristic function of $f(z)$ in the Ahlfors-Shimizu form.

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LEMMA 1. *Let $S(r,u) < \infty$, for any r, $0 < r < 1$. Then*

$$P(r,u) = \frac{1}{\pi} \iint\limits_{|z|<r} [u(z)]^2 \ln \frac{r}{|z|} dxdy, \quad z = x + iy,$$

for any r, $0 < r \leq 1$.

Lemma 1 is proved in [1].

For the Green potential

$$W(w) = \iint\limits_{|z|<1} \ln \left| \frac{1 - \bar{w}z}{z - w} \right| dxdy, \quad z = x + iy, \quad w \in D,$$

we proved in [2] the following lemma

LEMMA 2. *If $W(w)$ is a Green potential in D, then*

$$W(w) = \frac{1}{2}\pi(1 - |w|^2), \quad w \in D.$$

Let $\varphi_w(z) = \dfrac{z + w}{1 + \bar{w}z}$, $w \in D$ is fixed, and $u_w(z) = u(\varphi_w(z))|\varphi_w{}'(z)|$; obviously, $u_0(z) = u(z)$.

LEMMA 3. *Let $u(z) \geq 0$ be a measurable function in D. The following assertions are equivalent:*

$$\text{(i)} \qquad \iint\limits_{|z|<1} (1 - |z|)[u(z)]^2 \, dxdy < \infty, \quad z = x + iy$$

$$\text{(ii)} \qquad \iint\limits_{|w|<1} P(1, u_w) d\xi d\zeta < \infty, \quad w = \xi + i\zeta.$$

Lemma 3 is proved in [1].

For a measurable function $u(z) \geq 0$ defined in the unit disc D we introduce the differentiable form $d\mu_u(z) = (1 - |z|^2)[u(z)]^2 \, dxdy$, $z = x + iy$, and the measure $\mu_u(E) = \iint\limits_{E} d\mu_u(z)$ generated by $d\mu_u(z)$ on a Borel set $E \subset D$. Let

$$Q(\mu_u, w) = \frac{1}{2\pi} \mu_u(R(w)) \cdot (1 - |w|)^{-1},$$

where $R(w) = \{z \in D;\ |w| < |z| < 1,\ |\arg z - \arg w| < \pi \cdot (1 - |w|)\}$ for $w \neq 0$, and $R(w) = D$ for $w = 0$. The measure μ_u is called the Carleson measure if $\sup\limits_{w \in D} Q(\mu_u, w) < \infty$ (cf. [6]).

LEMMA 4. *The measure μ_u generated by the differentiable form $d\mu_u(z) = (1-|z|)[u(z)]^2\,dxdy$, $z = x+iy$, $z \in D$, is the Carleson measure if and only if*

$$\sup_{|\xi|<1} \iint_{|w|<1} P(u_w, 1)\cdot|\varphi_\xi{}'(w)|dudv < \infty, \quad w = u+iv$$

Lemma 4 is proved in [4].

2. A meromorphic function $f(z)$ defined in D belongs to the class BC, if $T(1,f) < \infty$.

THEOREM 1. *For a meromorphic function $f(z)$ in D and for any p, $0 < p < \infty$, the following assertions are equivalent:*

(i) $f(z) \in BC$;

(ii) $\displaystyle\iint_{|w|<1} P(1, (f_p^{\#})_w)dudv < \infty, \quad w = u+iv.$

Theorem 1 is proved in [1].

Following S.Yamashita, [7], a meromorphic function $f(z)$ defined in D is called a function with uniformly bounded characteristic if $\sup_{w\in D} T(1, f_w) < \infty$; $f(z)$ belong to the class UBC.

THEOREM 2. *For a meromorphic function $f(z)$ in D the following assertions are equivalent:*

(i) $f(z) \in UBC$

(ii) *The measure μ_f, μ_f generated by the differentiable form $d\mu_u(z) = (1-|z|^2)[f^{\#}(z)]^2\,dxdy$, $z = x+iy$, is Carleson measure;*

(iii) $\displaystyle\sup_{w\in D} \iint_{|z|<1} T(1, f_z)|\varphi_w{}'(z)|dxdy < \infty, \quad z = x+iy.$

Theorem 2 is proved in [3].

3. If $f(z)$ is a holomorphic function in D, we put $f_p^*(z) = \frac{1}{2}p|f(z)|^{\frac{p}{2}-1}|f'(z)|$, $0 < p < \infty$. Then $0 \le f_p^*(z) \le \infty$ and $f_p^*(z) = \infty$ at the zeros of $f(z)$. If $p = 2$, then $f_p^*(z) = |f'(z)|$ (cf. [8]).

THEOREM 3. *For a holomorphic function $f(z)$ in D and for any p, $0 < p < \infty$, the following assertions are equivalent:*

(i) *$f(z)$ belongs to the Hardy class H^p;*

(ii) $P(1, f_p^*) < \infty$;

(iii) $\displaystyle\iint_{|w|<1} P(1, (f_p^*)_w)dudv < \infty, \quad w = u+iv.$

Theorem 3 is proved in [1].

4. Let B denote the class of holomorphic functions $f(z)$ in D for which $|f(z)| < 1$ in D. For a functions $f(z) \in B$, let $f^h(z)$ denote the hyperbolic derivative of $f(z)$; i.e. $f^h(z) = |f'(z)|(1 - |f(z)|)^{-1}$. Consider $\lambda(f(z)) = \lambda(f) = -\ln(1 - |f(z)|)$.

Following S.Yamashita [9], we say that a function $f(z) \in B$ belongs to the hyperbolic Hardy class H_h^p, $0 < p < \infty$; if

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} (\sigma(f(z)))^p \, d\theta < \infty, \quad z = re^{i\theta},$$

where $\sigma(f(z)) = \dfrac{1}{2} \ln \dfrac{1 + |f(z)|}{1 - |f(z)|}$.

THEOREM 4. *For any function $f(z) \in B$ and for any p, $0 < p < \infty$, the following assertions are equivalent:*

(i) $f(z) \in H_h^p$;

(ii) $P(1, \lambda(f)^{(p-1)/2} f^h) < \infty$;

(iii) $\displaystyle\iint_{|w|<1} P(1, (\lambda(f)^{(p-1)/2} f^h)_w) dudv < \infty, \quad w = u + iv.$

Theorem 4 is proved in [1].

5. Let $f(e^{i\theta})$ be a summable function on the circumference $\Gamma : |z| = 1$. We denote by $f(z)$, $z \in D$ the Poisson integral generated by $f(e^{i\theta})$. Let $\nabla f(z)$ denote the gradient of the function $f(z)$.

THEOREM5. *For any summable function $f(e^{i\theta})$ on Γ the following assertions are equivalent:*

(i) $f(e^{i\theta}) \in BMO$;

(ii) $\sup_{w \in D} P(|\nabla f|_w, 1) < \infty$;

(iii) $\displaystyle\sup_{\xi \in D} \iint_{|w|<1} P(|\nabla f(z)|_w, 1) \, |\varphi_\xi{}'(w)|^2 \, dudv < \infty, \quad w = u + iv.$

THEOREM 6. *For a holomorphic function $f(z)$ in D the following assertions are equivalent:*

(i) $f(z) \in BMOA$;

(ii) $\sup_{w \in D} P(|f'|_w, 1) < \infty$;

(iii) $\sup_{\xi \in D} \iint_{|w|<1} P(|f'|_w, 1) |\varphi_\xi{}'(w)|^2 dudv < \infty, \quad w = u + iv$;

(iv) $\iint_{|w|<1} P(|gf'|_w, 1) dudv \leq c\|g\|^2, \quad w = u + iv$, *for any function $g(z)$ in the Hardy class H^2 with* $\|g\|^2 = \sup_{0<r<1} \int_0^{2\pi} |g(e^{i\theta})|^2 \, d\theta$.

Theorem 5 and theorem 6 are proved in [4].

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