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ON CERTAIN DIFFERENTIAL OPERATORS AND SOME CLASSES OF UNIVALENT FUNCTIONS

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ABSTRACT. *Using the differential operator defined by* $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z), \dots, D^n f(z) = D(D^{n-1} f(z))$, *we introduce new classes of univalent functions in the unit disc and consider some properties of them.*

1. Introduction. Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

For $f \in A$ we define

$$\begin{aligned} &D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z), \dots, \dots \\ &D^n f(z) = D(D^{n-1} f(z)), \quad n \in N = \{1, 2, \dots\}. \end{aligned} \tag{2}$$

We note that if f is defined by (1), then

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

Using the operator D^n, Salagean [6] considered the classes $S_n(\alpha)$ of functions $f \in A$ satisfying

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \alpha \quad (n \in N_0 = N \cup \{0\})$$

for some α $(0 \le \alpha < 1)$ and for all $z \in U$. Some other results on those classes were given in [4], too.

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Let O_n, $n \in N_0$, denote the class of functions $f \in A$ satisfying the condition

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} < \frac{n+2}{n+1}, \qquad z \in U. \tag{3}$$

In the second part of this paper it is proved that $O_{n+1} \subset O_n$, $n \in N_0$, and that O_n, $n \in N$, is a subclass of the class univalent functions in the unit disc. Certain integral transformation and subclasses of univalent functions with negative coefficients will be considered.

Let f and g be analytic functions in U. We say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there exists a function ω analytic in U such that $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in U$, and $f(z) = g(\omega(z))$.

2. Some properties of the classes O_n. In the beginning, we cite the following lemma whose more general form may be found in [2].

LEMMA 1 *Let p be analytic in U and q be analytic and univalent in $\overline{U}$ with $p(0) = q(0)$. If p is not subordinate to q then there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$, and a $k \geq 1$ for which $p(\,|z| < |z_0|\,) \subset q(U)$,*

$$\text{(i)} \qquad p(z_0) = q(\zeta_0), \quad \textit{and}$$
$$\text{(ii)} \qquad z_0 p'(z_0) = k\,\zeta_0\, q'(\zeta_0).$$

Also, we note that for $q(z) = z$ in the previous lemma we get Jack's well-known lemma [1], which we will use in the proof of the next theorem.

THEOREM 1 *Let $f \in O_{n+1}\,(n \in N_0)$. Then*

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec \frac{1-z}{1-az}, \tag{4}$$

where

$$a = \frac{3n+4}{3n+8} \tag{5}$$

PROOF. Let's put

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1-\omega(z)}{1-a\omega(z)}, \tag{6}$$

where a is defined by (5). Then $\omega(z)$ is analytic in U and $\omega(0) = 0$. From (6), after logarithmic differentiation and using the identity $z(D^m f(z))' = D^{m+1}f(z)$, $m \in N_0$, we get

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^n f(z)} = -\frac{z\omega'(z)}{1-\omega(z)} + \frac{az\omega'(z)}{1-a\omega(z)},$$

or using (6) once again

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = \frac{1-\omega(z)}{1-a\omega(z)} - \frac{z\omega'(z)}{1-\omega(z)} + \frac{az\omega'(z)}{1-a\omega(z)}. \tag{7}$$

Let's prove that $|\omega(z)| < 1$, $z \in U$. If not, then by Lemma 1 (where p is equal ω and $q(z) = z$) there exists z_0, $|z_0| < 1$, such that $|\omega(z_0)| = 1$, i.e. $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0) = ke^{i\theta}$, $k \geq 1$. For such z_0, from (7) and since $1/2 \leq a < 1$, we have

$$\begin{aligned}\operatorname{Re}\left\{\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)}\right\} &= \operatorname{Re}\left\{\frac{1-e^{i\theta}}{1-ae^{i\theta}} - \frac{ke^{i\theta}}{1-e^{i\theta}} + \frac{ake^{i\theta}}{1-ae^{i\theta}}\right\} \\ &= \frac{1+a}{2a} + \frac{2a-1}{2a} \cdot \frac{1-a^2}{1-2a\cos\theta + a^2} \\ &+ (k-1)\frac{1-a^2}{2(1-2a\cos\theta+a^2)} \\ &\geq \frac{1+a}{2a} + \frac{2a-1}{2a} \cdot \frac{1-a^2}{(1+a)^2} \\ &= \frac{5-a}{2(1+a)} \\ &= \frac{n+3}{n+2},\end{aligned}$$

which is a contradiction to $f \in O_{n+1} \Leftrightarrow \operatorname{Re}\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\} < \frac{n+3}{n+2}$ (by definition (3)). Therefore, $|\omega(z)| < 1$, $z \in U$, and by (6) we conclude that the relation (4) is valid.

From Theorem 1 we derive

THEOREM 2. *$O_{n+1} \subset O_n$ holds for every $n \in N_0$, and O_n, $n \in N$, are subclasses of the class of starlike functions in U.*

PROOF. If $f \in O_{n+1}$, $n \in N_0$, then from Theorem 1 we have

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} < \frac{2}{1+a} = \frac{3n+8}{3n+6} < \frac{n+2}{n+1},\ z \in U,$$

which implies $f \in O_n$. From Theorem 1 we also have that if $f \in O_1$ then

$$\frac{D^1 f(z)}{D^0 f(z)} = \frac{zf'(z)}{f(z)} \prec \frac{1-z}{1-\frac{z}{2}},$$

i.e. f is starlike in U. Since O_1 is the subclass of starlike functions in U, and $O_{n+1} \subset O_n \subset O_1$ $(n = 2, 3, \ldots)$, then the same is valid for the classes O_n, $n \geq 2$. We note that the class O_1 is equivalent to the class of functions $f \in A$ satisfying the condition $\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{3}{2}$, $z \in U$. The class O_1 was considered earlier in [5],[9] and [8]. The method given here is similar as in [3].

THEOREM 3 *Let $f \in O_n$, $n \in N_0$. Then the function*

$$F(z) = \frac{n+1}{z^n} \int_0^z t^{n-1} f(t)dt \tag{8}$$

also belongs to O_n.

PROOF. From (8) after differentiation we get

$$nF(z) + zF'(z) = (n+1)f(z),$$

and from there, applying the operator D^n,

$$nD^n F(z) + D^{n+1} F(z) = (n+1)D^n f(z).$$

From the last relation, after logarithmic differentiation, we obtain

$$\frac{nD^{n+1}F(z) + D^{n+2}F(z)}{nD^n F(z) + D^{n+1}F(z)} = \frac{D^{n+1}f(z)}{D^n f(z)} \tag{9}$$

If we put

$$p(z) = \frac{D^{n+1}F(z)}{D^n F(z)}, \quad q(z) = \frac{n+2}{n+1} - \frac{1}{n+1} \cdot \frac{1+z}{1-z}, \tag{10}$$

and since $\dfrac{D^{n+1}f(z)}{D^n f(z)} \prec q(z)$ on supposition in Theorem, from (9) we easily obtain

$$p(z) + \frac{zp'(z)}{n+p(z)} \prec q(z). \tag{11}$$

We want to prove that (11) implies $p(z) \prec q(z)$. In our case the function q defined by (10) has a simple pole at $z = 1$ on ∂U, but Lemma 1 is also true for such q (see [2], Lemma 1). If p is not subordinate to q, then by Lemma 1 there exists z_0, $|z_0| < 1$, and ζ_0, $|\zeta_0| = 1$ and $k \geq 1$ for which $p(|z| < |z_0|) \subset q(U)$ and the conditions (i) and (ii) of Lemma 1 are satisfied. If we put $\zeta_0 = e^{i\theta}$, then we have that the conditions (i) and (ii) give

$$p(z_0) = q(\zeta_0) = \frac{n+2}{n+1} - \frac{1}{n+1} i \cot \frac{\theta}{2},$$

and

$$z_0 p'(z_0) = \zeta_0 q'(\zeta_0) = \frac{1}{2(n+1)\sin^2 \frac{\theta}{2}}$$

For the same z_0 we easily get

$$\operatorname{Re}\left\{p(z_0)+\frac{z_0p'(z_0)}{n+p(z_0)}\right\}=\frac{n+2}{n+1}+\frac{1}{2\sin^2\frac{\theta}{2}}\cdot\frac{(n+1)^2+1}{\left|(n+1)^2+1-i\cot\frac{\theta}{2}\right|^2}$$
$$>\frac{n+2}{n+1},$$

which is a contradiction to (11). It follows that $p(z)\prec q(z)$, or $\operatorname{Re}\frac{D^{n+1}F(z)}{D^nF(z)}<\frac{n+2}{n+1}$, i.e. that $F\in O_n$, which was to be proved.

Finally, we may consider the class $\overline{O}_n$ of analytic functions in U which have the form

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k,\qquad a_k\geq 0, \tag{12}$$

and satisfy the condition (3). Independently of the result of Theorem 2, we can show that $\overline{O}_n$, $n\in N$, is a subclass of starlike functions in U. Namely, we have

THEOREM 4 *Let $f\in\overline{O}_n$, $n\in N$. Then*

$$\sum_{k=2}^{\infty}k^na_k\leq 1, \tag{13}$$

and f is starlike in U.

PROOF. Since $f\in\overline{O}_n$, $n\in N$, then the condition (3) implies that the function $\frac{D^{n+1}f(z)}{D^nf(z)}$ is subordinate to the function q defined in (10), i.e. that

$$\frac{D^{n+1}f(z)}{D^nf(z)}=\frac{n+2}{n+1}-\frac{1}{n+1}\cdot\frac{1+\omega(z)}{1-\omega(z)}, \tag{14}$$

where ω is analytic in U with $\omega(0)=0$ and $|\omega(z)|<1$, $z\in U$. From (14) we get

$$\omega(z)=\frac{D^{n+1}f(z)-D^nf(z)}{D^{n+1}f(z)-\frac{n+3}{n+1}D^nf(z)}. \tag{15}$$

Since $D^nf(z)=z-\sum\limits_{k=2}^{\infty}k^na_kz^k$ and $|\omega(z)|<1$, $z\in U$, then from (15) we have

$$|\omega(z)|=\left|\frac{\sum\limits_{k=2}^{\infty}(k-1)k^na_kz^{k-1}}{\frac{2}{n+1}+\sum\limits_{k=2}^{\infty}\left(k-\frac{n+3}{n+1}\right)k^na_kz^{k-1}}\right|<1,\quad z\in U.$$

which gives

$$\text{(16)} \qquad \operatorname{Re}\left\{ \frac{\sum_{k=2}^{\infty}(k-1)k^n a_k z^{k-1}}{\frac{2}{n+1}+\sum_{k=2}^{\infty}(k-\frac{n+3}{n+1})k^n a_k z^{k-1}} \right\} < 1, \quad z \in U.$$

If we take $z = r$, $0 \le r < 1$, then from (16) we have

$$\sum_{k=2}^{\infty} k^n a_k r^{k-1} \le 1,$$

and letting $r \to 1$ we get the relation (13).
Now, since

$$\sum_{k=2}^{\infty} k a_k \le \sum_{k=2}^{\infty} k^n a_k \le 1,$$

for $n \in N$, then f is starlike in U (see [7]).

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