Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 6 (1992), 101–105

FILOMAT–20, Niš, September 26–28, 1991

THE CARLESON MEASURE AND DIFFERENTIABLE FORMS IN THE UNIT DISC

Novo Labudović

Abstract. *Reling on the results given in* [1, 2, 4, 8 *and* 9], *we have investigated the Carleson measure and differentiable forms and we give sufficient conditions for uniformly bounded Nevanlinna-Ahlfors-Shimizu characteristics. Also, we give a new characterisation of the class* $BMOA_\sigma$.

1. Let $g(z)$ be a measurable function defined in the unit disc $D : |z| < 1$ in the complex z-plane , and let $g(z) \geq 0$ for any $z \in D$. For a point $w \in D$, we put $\varphi_w(z) = (z+w)(1+\bar{w}z)^{-1}$ and $g_w(z) = g(\varphi_w(z)) \cdot |\varphi'_w(z)|$; $z \in D$.

We introduce the characteristic function $P(g,r)$ for the function $g(z)$, as follows

$$P(g,r) = \int_0^r \frac{S(t,g)}{t}\,dt\,, \qquad 0 < r < 1$$

where

$$S(g,t) = \frac{1}{\pi} \iint\limits_{|z|<t} [g(z)]^2\,dxdy\,, \qquad z = x+iy.$$

Let $f(z)$ be a meromorphic function in D and let $f^{\#}(z)$ denote the spherical derivate of $f(z)$; that is $f^{\#}(z) = |f'(z)|(1+|f(z)|^2)^{-1}$, $z \in D$. If $g(z) = f^{\#}(z)$, then $P(f^{\#}(z), r) = T(f,r)$ become the Nevanlinna characteristic function in the form of Ahlfors-Shimizu for the meromorphic function f(z).

The function P(g,r) , $0 < r \leq 1$, is called the Nevanlinna-Ahlfors characteristic function for g(z).

The following result is proved in [1].

Theorem A. *If* $S(g,r) < \infty$ *for any* r, $0 < r < 1$, *then*

$$P(g,r) = \frac{1}{\pi} \iint\limits_{|z|<r} [g(z)]^2 \ln\frac{r}{|z|}\,dxdy, \qquad z = x+iy,$$

for any r, $0 < r \leq 1$.

We say for a measurable function $g(z) \geq 0$ in D to have uniformly bounded Nevanlinna-Ahlfors-Shimizu characteristic if $\sup_{w \in D} P(g_w, 1) < \infty$

2. A measure λ defined in D is called the Carleson measure, if there exists a constant $c = c(\lambda), 0 < c < \infty$, such that $\lambda(S) = c \cdot h$ for any set $S = \{z = re^{i\theta} \in D; 1 - h < r < 1, |\theta - \theta_0| \leq h\}, \quad 0 < h < 1$.

The following result is contained in [2].

THEOREM B. *A measure λ in D is the Carlesan measure if and only if*

$$\sup_{w \in D} \iint\limits_{|z|<1} \frac{1 - |w|^2}{|1 - \bar{w}z|^2} \, d\lambda(z) = M < \infty.$$

3. For a measurable function $g(z) \geq 0$ defined in D, we introduce the defferentiable form $d\lambda_g = (1 - |z|^2) [g(z)]^2 \, dxdy, \quad z = x + iy$, and the measure $\lambda_g(E) = \iint\limits_E d\lambda_g$ generated by $d\lambda_g$, for a Borel set $E \subset D$.

THEOREM 1. *If a measurable function $g(z) \geq 0$ defined in D, satisfies the conditions :*

(i) *λ_g is the Carleson measure in D ;*

(ii) $\sup_{z \in D} (1 - |z|^2) \cdot g(z) = c < \infty$,

then $g(z)$ has the uniformly bounded Nevanlinna-Ahlfors-Shimizu characteristic.

PROOF. Since λ_g is the Carleson measure, it follows from Theorem B, that

$$\sup_{w \in D} \iint\limits_{|z|<1} \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \bar{w}z|^2} [g(z)]^2 \, dxdy = c_1 < \infty. \tag{1}$$

Using (1) and the inequality $-\ln t \leq \frac{1}{2c_2{}^2}(1 - t^2)$, which is valid for some c_2, $0 < c_2 < \infty$, and any t, $c_2 < t < 1$, we obtain

$$\begin{aligned} \iint\limits_{D \setminus \nabla(w,c_2)} [g(z)]^2 \cdot \ln \left| \frac{1 - \bar{w}z}{z - w} \right| dxdy &\leq \frac{1}{2c_2{}^2} \iint\limits_D [g(z)]^2 \left(1 - \left| \frac{z - w}{1 - \bar{w}z} \right|^2\right) dxdy = \\ \frac{1}{2c_2{}^2} \iint\limits_D [g(z)]^2 \, &\frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \bar{w}z|^2} \, dxdy < \frac{c_1}{2c_2{}^2} \end{aligned} \tag{2}$$

where $w \in D$ and $\Delta(w, c_2) = \{z \in D; \left| \frac{z - w}{1 - \bar{w}z} \right| < c_2\}$.

Since $(1 - |z|^2) g(z) \leq c < \infty$, for any $z \in D$, then the estimate

$$\iint\limits_{\Delta(w,c_2)} [g(z)]^2 \cdot \ln \left| \frac{1 - \bar{w}z}{z - w} \right| dxdy \leq c^2 \iint\limits_{\Delta(w,c_2)} (1 - |z|^2)^{-2} \cdot \ln \left| \frac{1 - \bar{w}z}{z - w} \right| dxdy, \tag{3}$$

$z = x + iy$, is valid for any $w \in D$.

Setting $\xi = \dfrac{z-w}{1-\bar{w}z}, \xi = u + iv$ in (3) , we get

$$\iint\limits_{\Delta(w,c_2)} [g(z)]^2 \cdot \ln\left|\frac{1-\bar{w}z}{z-w}\right| dxdy \le c^2 \iint\limits_{|\xi|<c_2} \left(1-|\xi|^2\right)^{-2} \ln\frac{1}{|\xi|}\, dudv \le c_3 < \infty. \tag{4}$$

It follows from (2) and (4) , that

$$\frac{1}{\pi} \iint\limits_{D} [g(z)]^2 \cdot \ln\left|\frac{1-\bar{w}z}{z-w}\right| dxdy \le M < \infty \tag{5}$$

for any $w \in D$.

It follows from Theorem A, that

$$P(1, g_w) = \frac{1}{\pi} \iint\limits_{D} [g(z)]^2 \cdot \ln\left|\frac{1-\bar{w}z}{z-w}\right| dxdy \le M < \infty \tag{6}$$

$z = x + iy$.

Combing (5) and (6) , we obtain the inequality $\sup\limits_{w\in D} P(1, g_w) \le M$ which proves Theorem 1.

4. The following result is proved in [4].

THEOREM C. *The measure* $\lambda_g = (1-|z|^2)[g(z)]^2\, dxdy$, $z = x + iy$, $z \in D$, *is the Carleson measure if and only if*

$$\sup_{w\in D} \iint\limits_{|z|<1} P(1, g_w) \cdot |\varphi'(z)|^2\, dxdy < \infty \quad , z = x + iy.$$

In particular, if a measurable function $g(z) \ge 0$ in D has the uniformly bounded Nevanlinna-Ahlfors-Shimizu characteristic, then the measure λ_g generated by the differentiable form

$$d\lambda_g(z) = (1-|z|^2)[g(z)]^2\, dxdy, \quad z \in D, \quad z = x + iy,$$

is the Carleson measure.

THEOREM 2. *Let* $g(z) \ge 0$ *be a measurable function in D. Denote* $r(a) = 1/5(4a-1)$, $a \in (1/4, 1)$. *Then for any* w, $a < |w| < 1$, *the estimate*

$$T(r(a), g_w) = c \cdot \sup_{w\in D} \iint\limits_{|z|<1} [g(z)]^2 \frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{w}z|^2}\, dxdy, \quad z = x + iy,$$

is valid for some constant c, $0 < c < \infty$.

PROOF. It follows from Theorem A that

$$T(r(a), g_w) = \frac{1}{\pi} \iint\limits_{|z|<1} [g(z)]^2 \frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{w}z|^2} \, dxdy, \quad z = x + iy, \tag{8}$$

for any $w \in D$.

Since in $\ln \frac{1}{|z|} \leq c \cdot (1 - |z|^2)$ for any $|z| > \frac{1}{4}$ for some c, $0 < c < \infty$ and $\frac{1}{4} < |\frac{z-w}{1-\bar{w}z}|$ for any z, $|z| < r(a)$, and any w, $a < |w| < 1$ where $\frac{1}{4} < a < 1$, then

$$\ln |\frac{1-\bar{w}z}{z-w}| \leq c(1 - |\frac{1-\bar{w}z}{z-w}|^2) = c\frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{w}z|^2} \tag{9}$$

for any z, $|z| < r(a)$, and any w, $a < |w| < 1$, with $\frac{1}{4} < a < 1$.

Combining (8) and (9), we get (7) , and Theorem 2 is proved.

6. A meromorphic function $f(z)$ defined in disc D is said to be in the class UBC of functions with the uniformly bounded Nevanlinna characteristic if $\sup_{w \in D} T(1, f_w) < \infty$.

This class was studied in [1], [3], [5], [6] .

Generating the Carleson measure, $f(z)$ belongs to the class UBC.

We note that this result can be obtained from Theorem 1 by setting $g(z) = f^{\#}(z)$.

In [3] S.Yamashita has posed the problem: Does a meromorphic function $f(z)$ belong to the class UBC, if the differentiable form $d\lambda_f = (1-|z|^2)[f(z)]^2 \, dxdy$, $z = x + iy$ generates the Carleson measure ?

The solution of problem in the affirmative was given by Ž. Pavićević in [6].

The solution can be obtained from Theorem 2 and the following result of S. Yamashita [8].

THEOREM D. *If $f(z)$ is meromorphic function in D, then* $(1-|z|^2) \cdot [f^{\#}(z)]^2 \leq \frac{1}{r^2}(\exp(2T(r, f_w) - 1)$ *for any* $w \in D$ *and any* r, $0 < r < 1$.

7. A holomorphic function $f(z)$ in D belongs to the class B if $|f(z)| < 1$ for any $z \in D$.

For function $f(z) \in B$, let $f^h(z) = |f'(z)|[1 - |f(z)|^2]^{-1}$ be the hyperbolic derivative of $f(z)$. The function $T^h(f, r) = P(f^h, r)$, $0 < r \leq 1$, is called the hyperbolic Nevanlinna-Ahlfors-Shimizu characteristic function for $f(z) \in B$.

In [9] S.Yamashita applied the function $T^h(f, r)$ to define and investigate the functional class $BMOA_\sigma$. A holomorphic function $f(z) \in B$ belongs to the class $BMOA_\sigma$ if $\sup_{w \in D} T^h(f_w, 1) < \infty$.

THEOREM 3. *A holomorphic function $f(z) \in B$ belongs to the class $BMOA_\sigma$ if and only if the differentiable form $d\lambda_f = (1-|z|^2)[f^h(z)]^2\, dxdy$, $z = x+iy$ generates the Carleson measure.*

The assertion of Theorem 3 is a consequence of Theorem 1 and the following result of S.Yamashita, [8], p.195.

THEOREM E. *If $f(z) \in B$, then inequality*

$$(1-|z|^2)[f^h(z)]^2 \leq \frac{1}{r^2}(1-\exp[-2T^h(f_w, r)])$$

holds for any $w \in D$ and any r, $0 < r < 1$.

Combing Theorem 3 and Theorem C, we get

THEOREM 4. *A holomorphic function $f(z) \in B$ belongs to the class $BMOA_\sigma$ if and only if*

$$\sup_{w\in D} \iint\limits_{|z|<1} T(f_z^h, 1) \cdot |\varphi_w'(z)|^2\, dxdy < \infty, \quad z = x+iy.$$

REFERENCES

[1] PAVIĆEVIĆ Ž., *Characteristic properties of the Nevanlinna class N and the Hardy classes H^p and H_h^p*, Anal. Acad. Scient. Fenice, ser. AI, Math., 15 (1990) , 3-9.

[2] GARNATT B.J., *"Bounded analytic function"* Academic Press, 1981.

[3] YAMASHITA S., *Image area and functions of uniformly bounded characteristic*, Comm. Math. Univ. Sanc. Pauly, 34(1) (1985), 37-44.

[4] PAVIĆEVIĆ Ž., *The Carleson measure, Ahlfors-Shimizu characteristic and classes BMO and $BMOA$* Dokl. Akad. Nauk SSSR, 317(2) (1991), 309-312.

[5] YAMASHITA S., *Functions of uniformly bounded characteristic* Ann. Acade. Scient. Fenice, seria AI, Math., 7 (1982), 349-367.

[6] PAVIĆEVIĆ Ž., *The Carleson measure and meromorphic functions of uniformly bounded characteristic* Ann. Acade. Scient. Fenice, seria AI, Math., 16 (1991) , (to appear).

[7] YAMASHITA S., *Some unsolved problems of meromorphic functions of uniformly bounded characteristic* Inter. J. Math. Sci., 8(3) (1985), 477-482.

[8] YAMASHITA S., *Estimates of derivatives by the Shimizu-Ahlfors characteristic functions*, Comm. Math. Univ. Sanc. Pauly , 35(2) (1986), 189-199.

[9] YAMASHITA S., *F.Riez's decomposition of a subharmonic function, applied to $BMOA$*, Bollettion U.M.I., 6(3-A) (1984), 103-109.

[10] LABUDOVIĆ N. PAVIĆEVIĆ Ž., *Hiperbolic and parabolic Bloch functions*, Mat. Vesnik , 40 (1988) , 275-279 (Russian).

University of Podgorica
Faculty of Mathematical Sciences
Cetinjski put b.b.
81 000 Podgorica
Yugoslavia