Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 6 (1992), 1–3

**FILOMAT–20**, Niš, September 26–28, 1991

# AN EXTENSION OF THE SION'S MINIMAX THEOREM

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ABSTRACT. *In this paper we extend the Sion's minimax theorem, omitting hypotheses of convexity for coordinate functions.*

Let $X$ and $Y$ are non empty sets. A function $f : X \times Y \mapsto R$ has a saddle point $(x_0, y_0) \in X \times Y$ if:

$$\max_{x\in X} \min_{y\in Y} f(x,y) = \min_{y\in Y} \max_{x\in X} f(x,y) = f(x_0, y_0)\,.$$

Theorems about existence of saddle points, so called minimax theorems, are especially important in games theory. Sion [3] has proved one of the most general theorem, of this type. Tasković [4] has got a new proof of Sion's theorem.

For a nonempty set $X$, function $g : X \mapsto R$ and real number $r \in R$ set

$$\underline{L}(g,r) = \{\, x \in X \,:\, g(x) < r \,\} \quad \text{and} \quad \overline{L}(g,r) = \{\, x \in X \,:\, g(x) > r \,\}\,.$$

If $X$ is a topological space, then $g$ is a lower ( upper ) semicontinuous if $\overline{L}(g,r)$ ( $\underline{L}(g,r)$ ) is open set for each $r \in R$. If $g$ is a lower ( upper ) semicontinuous function, than $g$ has a minimum ( maximum ) on compact sets. If $X$ is a convex subset of some topological vector space, then $g$ is quasi-concave ( quasi-convex ) if $\overline{L}(g,r)$ ( $\underline{L}(g,r)$ ) is a convex set for each $r \in R$.

Sion [3] has proved the following minimax theorem.

THEOREM 1 *Let $X \subseteq E$ and $Y \subseteq F$ are nonempty compact convex sets in topological vector spaces $E$ and $F$. If $f : X \times Y \mapsto R$ is a real-valued function such that:*

a) $y \mapsto f(x,y)$ *is lower semicontinuous and quasi-convex on $Y$ for each $x \in X$;*

b) $x \mapsto f(x,y)$ *is upper semicontinuous and quasi-concave on $X$ for each $y \in Y$,*

*then $f$ has a saddle point.*

The aim of this note is to prove Theorem 1, without the assumption of convexity for coordinate functions. ( See also Simons [2] and Tasković [4] ).

Recall that Ky Fan [1] has proved the following coincidence theorem.

THEOREM 2 *Let $X \subseteq E$ and $Y \subseteq F$ are two nonempty compact convex sets in topological vector spaces $E$ and $F$. Let $A, B : X \mapsto 2^Y$ satisfy:*

a) *$A(x)$ is open and $B(x)$ is nonempty convex set for $x \in X$;*
b) *$B^{-1}(x)$ is open and $A^{-1}(y)$ is nonempty convex set for each $y \in Y$.*
*Then $A(x_0) \cap B(x_0) \neq \emptyset$ for some $x_0 \in X$.*

Now we prove our

THEOREM 3 *Let $X \subseteq E$ and $Y \subseteq F$ are nonempty compact convex sets in topological vector spaces $E$ and $F$. If $f : X \times Y \mapsto R$ is a real-valued function such that*
a) *there is $T \subseteq R$ such that $a, b \in f(X \times Y)$, $a < b$ implies $T \cap (a,b) \neq \emptyset$;*
b) *function $f_x(y) = f(x,y)$ is lower semicontinuous on $Y$, and $L(f_x, t)$ is convex for each $t \in T$, $x \in X$;*
c) *function $f_y(x) = f(x,y)$ is upper semicontinuous on $X$, and $L(f_y, t)$ is convex for each $t \in T$, $y \in Y$,*

*then $f$ has a saddle point.*

PROOF. Since $f$ is upper and lower semicontinuous, then $\min_{y \in Y} f(x,y) = p(x)$ and $q(y) = \max_{x \in X} f(x,y)$ exist for each $x \in X$ and $y \in Y$. Also, since $p$ is upper semicontinuous on $X$ and $q$ is lower semicontinuous on $Y$, we know that $\max_{x \in X} p(x)$ and $\min_{y \in Y} q(y)$ exist.

Inequalities,

$$p(x) \leq q(y) \quad \text{for each} \quad x \in X \text{and} \quad y \in Y,$$

implies

$$\max_{x \in X} p(x) \leq \min_{y \in Y} q(y).$$

If

$$\max_{x \in X} p(x) < \min_{y \in Y} q(y)$$

then there is $t \in T$, such that

$$\max_{x \in X} p(x) < t < \min_{y \in Y} q(y).$$

We define mappings $A, B : X \mapsto 2^Y$ by

$$A(x) = \{\, y : t < f(x,y) \,\} \quad \text{and} \quad B(x) = \{\, y : f(x,y) < t \,\}, \quad x \in X.$$

Since $f(x,y)$ is lower semicontinuous, we have that $A(x)$ is open for each $x \in X$. Further, since $\max_{x \in X} p(x) < t$, it follows that $B(x)$ is convex and nonempty for each $x \in X$.

$A^{-1}(y) = \{\, x : t < f(x,y) \,\}$ and $B^{-1}(y) = \{\, x : f(x,y) < t \,\}$ implies that $A^{-1}(y)$ is nonempty and convex, and $B^{-1}(y)$ is open. By Theorem 2 there is $x \in X$ such that $A(x_0) \cap B(x_0) \neq \emptyset$. This is a contradiction with $f(x_0, a) < t < f(x_0, b)$, for each $a \in A(x_0)$ and $b \in B(x_0)$, and so the proof is complete.

REMARK. We see, that in Theorem 1, we have $T = R$.

## References

[1] F. Ky, *Minimax theorems*, Proc. Nat. Sci. USA, 39, (1953), 42–47.

[2] Simons S., *An upward-doward minimax theorem*, Arch. Math., 54(3), (1990), 275–279.

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[4] Tasković M. R., *Minimax theorems on posets with applications*, Math. Japanica, 35(5), (1990), 805–815.

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