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FIBER SHAPE THEORY AND RESOLUTIONS

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Abstract. In this paper a fiber shape category is defined for all topological spaces over B. Our definition is based on the prohomotopy category of inverse systems of absolute fibrewise neighborhood retracts. A fundamental role is played by the resolutions of spaces over B.

1. Introduction

The notion of fiber shape, recently introduced by the author ([Ba_1], [Ba_2], [Ba_3]), M. Clapp and L. Montejeno ([CM]), H. Kato ([Ka]), S. C. Metcalf ([Me]), T. Yagasaki ([Ya]), is a modification of homotopy type of maps ([Sp])

The purpose of this paper is to exibit a notion of fiber shape for arbitrary topological spaces over a metrizable space B.

In our development we follow the method of ANR_B-space ([Ja], [Ya]). First, we prove that every topological space over a metrizable space B admits an ANR_B-resolution; second, we show that the homotopy category [ANR_B], which consists of spaces over B having the fiber homotopy types of ANR_B-s, is a dense full subcategory of the homotopy category [Top_B] of spaces over B.

The fiber shape category Sh_B is by definition the general shape category $\mathcal{S}h_{(\mathcal{T},\mathcal{P})}$ of S. Mardešić, where $\mathcal{T}$ = [Top_B] and $\mathcal{P}$ = [ANR_B]

The fiber shape category coincides with the fiber shape category of T. Yagasiki ([Ya]) for the category of metrizable spaces over B. Also note that if B is an one-point space, then $\mathcal{S}h_B$ coincides with the shape category $\mathcal{S}h$ of S. Mardešić ([MS]).

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The results, set forth in this paper, were reported at the International Topological Conference in Baku in 1987; brief reports on them have appeard in the author's papers ([Ba_1],[Ba_2],[Ba_3]

2. Notation and preliminaries

We use the following notation. Let B denote the fixed space. The space X over B is a pair consisting of a topological space X and a continuous mapping $\Pi_X: X \to B$. Let X and Y be spaces over B. A continuous map $f: X \longrightarrow Y$ is said to be a fiber preserving (f.p) if $\Pi_Y \circ f = \Pi_X$. By Top_B we denote the category of all spaces over B and all f.p. maps.

Two f.p. maps $f, g: X \longrightarrow Y$ of the category Top_B are said to be fiber preserving (f.p.) homotopic, $f \underset{B}{\simeq} g$, if there is a homotopy $H: X \times I \longrightarrow Y$ from f to g, such that $\Pi_Y \circ H = \Pi_{X\times I}$, where $\Pi_{X\times I}(x,t) = \Pi_X(x)$ for every $t \in I$ and $x \in X$. The homotopy H is called fiber preserving homotopy or homotopy over B. The relation $\underset{B}{\simeq}$ is an equivalence relation and we denote by $[f]_B$ the homotopy class of the f.p. map f. The relation $\underset{B}{\simeq}$ is compatible with the composition. Therefore, one can define the composition of class $[f]_B: X \longrightarrow Y$ and $[g]_B: Y \longrightarrow Z$ by composing representatives:

$$[g]_B \circ [f]_B = [g \circ f]_B.$$

$[Top_B]$ denotes the homotopy category of Top_B. Its objects are all the objects of Top_B and the morphisms are equivalence classes with respect to $_B\simeq$ of morphisms in Top_B. Two spaces over B X and Y are said to be fiber homotopy equivalent, $X \underset{B}{\simeq} Y$, if there exist two f.p. maps $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ such that $g \circ f \underset{B}{\simeq} 1_X$ and $f \circ g \underset{B}{\simeq} 1_Y$.

We denote by $C(I,X)$ the space of all continuous maps $\varphi: I \longrightarrow X$ endowed with the compact-open topology. $C_B(I,X)$ denotes the subspaces of $C(I,X)$ consisting of all continuous maps $\varphi: I \longrightarrow X$ such that $\pi_X \circ \varphi = \text{const}$.

3. Retracts and extensors of metrizable spaces over B

Let B be a fixed metrizable space and $\mathfrak{M}_B$ the category of all metrizable spaces over B and all f.p. maps.

Let X be a metrizable space over B and Y a subspace of X. A f.p. map $r: X \longrightarrow Y$ is called a fibrewise retraction if $r \circ i = 1_Y$, where $i: Y \longrightarrow Y$ is the f.p. inclusion map. In this case the subspace Y is called a fiber retract of X.

A subspace Y of metrizable space X over B is called a fibrewise neighborhood retract of X if there exist a neighborhood $\mathcal{U}$ of Y in X and a fibrewise retraction $r: \mathcal{U} \longrightarrow Y$.

The space $Y \in \mathfrak{M}_B$ is an absolute retract over B, $Y \in AR_B$ (an absolute neighborhood retract over B, $Y \in ANR_B$), if Y has the following property: for any closed f.p. embedding $i: Y \longrightarrow X \in \mathfrak{M}_B$ there exists a fibrewise retraction $r:X \longrightarrow i(Y)$ (a neighborhood $\mathcal{U}$ of $i(Y)$ in X and a fibrewise retraction $r:\mathcal{U} \longrightarrow i(Y)$).

The space $Y \in \mathfrak{M}_B$ is an absolute extensor over B, $Y \in AE_B$ (an absolute neighborhood extensor over B, $Y \in ANE_B$), if it has the following property: for any space $X \in \mathfrak{M}_B$ and any closed subset $A \subseteq X$, every f.p. map admits a f.p. extension $\underline{f}:X \longrightarrow Y$ ($\underline{f}:\mathcal{U} \longrightarrow Y$, where $\mathcal{U}$ is a neighborhood of A in X).

The following proposition is proved by T. Yagasaki ([Ya]), (see the proof of Proposition 1.3).

PROPOSITION 1.([Ya]) *A metrizable space Y over B is an ANR_B if and only if Y is an ANE_B.* ∎

PROPOSITION 2. (comp. [Ya], Proposition 1.1.) *For every metrizable space X over B there exists an ANE_B-space M over B with weight*

$$w(M) \le max\{w(X), w(B), \omega\}$$

and there exists a f.p. embedding $i:X \longrightarrow M$ such that $i(X)$ is closed in M.

Proof. By the Kuratowski-Wojdysławski embedding theorem for every metric space X there is a normed vector spae L an an embedding $h:X \longrightarrow L$ such that $h(X)$ is a closed subset of its convex hull $K \subseteq L$. We may assume that $w(K) \le max\{w(X), \omega\}$. By the Dugunji extension theorem every convex subset of a normed vector space is an AE for metric space. Every map $\Pi_X:X \longrightarrow B$ admits a closed embedding $i:X \longrightarrow BxK$ given by

$$i(X) = (\Pi_X(x), h(x)) \ , \ x \in X.$$

The space $M = BxK$ is an ANE_B-space over B ([Ya], see Proposition 1.1.). It is easy to see that $w(M) \le max\{w(X), w(\beta), \omega\}$. ∎

In ([Ya]) T. Yagasaki has proved the following proposition.

PROPOSITION 3.([Ya]) *Let* $p:X \longrightarrow B$ *by a map. If* $B = B_1 \cup B_2$, $B_i \subseteq B$ *closed and* $p_{B_i}(p|p^{-1}(B_i)):p^{-1}(B_i) \longrightarrow B_i$ *is an* ANR_{B_i} $(i = 1,2)$, *then* p *is an* ANR_B. *If each* $b \in B$ *admits a neighborhood* $\mathcal{U}$ *for which* $p_{\mathcal{U}}$ *is an ANR over* $\mathcal{U}$, *then* p *is an* ANR_B. ∎

Generalizing Propositions 8.1., 9.1., 10.1.,10.2. of ([Hu],ch.II) we have the following results.

PROPOSITION 4. (comp. [Ja], Proposition 8.21) *Let $Y = Y_1 \cup Y_2 \in \mathfrak{M}_B$. If Y_1 and Y_2 are open subset of Y and $Y_1 \cup Y_2 \in ANE_B$ ($Y_1, Y_2 \in AE_B$), then $Y \in ANE_B$ ($Y \in AE_B$).* ∎

PROPOSITION 5. (comp.[Fe-Ch], Proposition 2.9.) *Let $Y = Y_1 \cup Y_2 \in \mathfrak{M}_\beta$. If Y_1, Y_2 are closed subsets of Y and $Y, Y_1 \cap Y_2 \in ANE_B$ ($Y, Y_1 \cap Y_2 \in AE_B$), then $Y_1, Y_2 \in ANE_B$ ($Y_1, Y_2 \in AE_B$).* ∎

PROPOSITION 6. (comp. [Ja]), Proposition 8.23., Ch 8.) *Let* $Y = Y_1 \cup Y_2 \in \mathfrak{M}_B$. *If* Y_1, Y_2 *are closed subsets of* Y *and* $Y_1, Y_2, Y_1 \cap Y_2 \in ANE_B$ $(Y_1, Y_2, Y_1 \cap Y_2 \in AE_B)$, *then* $Y \in ANE_B$ $(Y \in AE_B)$. ∎

The proofs of Propositions 4, 5 and 6 are analogous to the proofs of the corresponding propositions in the clasical theory; therefore they are omitted.

Let $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in A}$ be a covering of a space Y. We say that the maps $f, g: X \longrightarrow Y$ are $\mathcal{U}$-near, if for every $x \in X$ there exists a $U_\alpha \in \mathcal{U}$ such that $f(x), g(x) \in U_\alpha$. We say that a homotopy $H: X x I \longrightarrow Y$ which connects f and g, is a $\mathcal{U}$-homotopy if for every $x \in X$ there exists a $u_\alpha \in \mathcal{U}$ such that $H(x,t) \subseteq U_\alpha$ for all $t \in I$.

PROPOSITION 7. (comp. [Fe-Ch], Proposition 1.2., Ch. III) *Let Y be a* ANR_B. *Then every open covering* $\mathcal{U}$ *of Y admits an open covering* $\mathcal{V}$ *of Y such that any two* $\mathcal{V}$*-near f.p. maps* $f, g: X \longrightarrow Y$ *from an arbitrary space X over* B *into the space Y over* B *are f.p.* $\mathcal{U}$*-homotopic. Moreover, if for a given* $x \in X$, $f(x) = g(x)$, *then* $H(x,t) = f(x)$ *for every* $t \in I (H: f \simeq_B g)$.

Proof. We may assume that Y is a closed subset of space $\beta x K$, where K is a convex set of normed vector space L. Let $\Pi: B x K \longrightarrow K$ be the map given by the formula $\Pi(b,k) = k$ for every $(b,k) \in B x K$. Since Y is an ANR_B, there is an open neighborhood G of Y in BxK together with a fibrewise retraction $r: G \longrightarrow Y$. Let $\{O_\mu \times Q_\mu\}_{\mu \in N}$ be a refinement of $r^{-1}(\mathcal{U})$, where Q_μ is convex for every $\mu \in N$. Then $\mathcal{V} = \{(O_\mu \times Q_\mu) \cap Y\}_{\mu \in N}$ is an open refinement of the covering $\mathcal{U}$. For any two $\mathcal{V}$-near f.p. maps $f, g: X \longrightarrow Y \subseteq B \times K$ we can define a f.p. homotopy $H': X x I \longrightarrow K$ by

$$H'(x,t) = (\Pi_X(x), (1-t)\Pi(f(x)) + t\Pi(g(x)), \quad (x,t) \in X x I.$$

Define a f.p. map $H: X x I \longrightarrow Y$ by taking

$$H(x,t) = r(H'(x,t)), \quad (x,t) \in X x I.$$

Clearly, we have $H_o = f$, $H_1 = g$, $H = \Pi_{XxI}$ and H is a $\mathcal{U}$-homotopy. Obviously, if $f(x) = g(x)$, then $H(x,t) = f(x) = g(x)$ for every $t \in I$. ∎

T. Yagasaki in ([Ya]) showed the following proposition.

PROPOSITION 8. ([Ya]) *Let* $Y \in ANR_B$. *Let A be a closed subspace of a metrizable space X over* B. *Let* $f, g: X \longrightarrow Y$ *be f.p. maps and let* $H: A x I \longrightarrow Y$ *be f.p. maps and let* $H: A x I \longrightarrow Y$ *be a homotopy over* B *from* $f|A$ *to* $g|A$. *Then there exists a neighborhood* $\mathcal{U}$ *of A in X and homotopy over* B, $H: \mathcal{U} x I \longrightarrow Y$ *from* $g|\mathcal{U}$ *to* $f|\mathcal{U}$. ∎

Let C(Z,Y) be the funcion space with the compact-open topology. It is known that if Z is a compact space, then the compact-open topology on $C(Z,Y)$ agrees with the topology induced by the metric:

$$d(f,g) = \sup\{d(f(z),g(z)) : z \in Z,\ f,g \in C(Z,Y)\}.$$

Consider the subspace $C_B(Z,Y)$ of the space $C(Z,Y)$:

$$C_B(Z,Y) = \{f \in C(Z,Y) : \Pi_Y \circ f = \text{const}\}.$$

Let $\Pi_{C_B(Z,Y)} : C_B(Z,Y) \longrightarrow B$ be a map given by $\Pi_{C_B(Z,Y)}(f) = \Pi_Y(f(z))$, $z \in Z$. Consequently, the pair consisting of the space $C_B(Z,Y)$ and the map $\Pi_{C_B(Z,Y)}$ is a space over B.

PROPOSITION 9. *Let Y be an ANE_B-space and let Z be a compact metric space. Then the space $C_B(Z,Y)$ is an ANE_B-space.*

Proof. Let X be a metric space over B, A closed subspace of X and $f : A \longrightarrow C_B(Z.Y)$ be a f.p. map. The map $F : A\times Z \longrightarrow Y$ given by $F(a,z) = (f(a))(z)$, $(a,z) \in A\times Z$ is a f.p. map. Indeed, for every $(a,z) \in A\times Z$ we have

$$(\Pi_Y \circ F)(a,z) = \Pi_Y(F(a,z)) = \Pi_Y((f(a)(z)) = \Pi_{C_B(Z,Y)}(f(a)) =$$

$$\Pi_{X|A}(a) = \Pi_{A\times Z}(a,z).$$

Observe that since $Y \in AE_B$, there exist a neighborhood $\mathcal{U}$ of $A\times Z$ in $X\times Z$ and a f.p. map $\tilde{F} : \mathcal{U} \longrightarrow Y$ such that $\tilde{F}|A\times Z = F$. Using the compactness of Z one can find a neighborhood $\mathcal{V}$ of A in X such that $\mathcal{V} \times Z \subseteq \mathcal{U}$.

We define $\tilde{f} : \mathcal{V} \longrightarrow C_B(Z,Y)$ to be the following: $(\tilde{f}(v))(z) = \tilde{F}(v,z)$, $v \in \mathcal{V}$, $z \in Z$. For each $a \in A$ we have $(\tilde{f}(a))(z) = \tilde{F}(a,z) = F(a,z) = (f(a))(z)$. Consequently, $\tilde{f}|A = f$. The map $\tilde{f} : \mathcal{V} \longrightarrow C_B(Z,Y)$ is a f.p. map. Indeed,

$$(\Pi_{C_B(Z,Y)} \circ \tilde{f})(v) = \Pi_{C_B(Z,Y)}(\tilde{f}(v)) = \Pi_Y((\tilde{f}(v))(z) = \Pi_Y(\tilde{F}(v,z)) =$$

$$\Pi_{X\times Z|\mathcal{V}\times Z}(v,z) = \Pi_{X|\mathcal{V}}(v).$$

This completes the proof. ∎

These propositions are used in section 4 and 5.

4. Resolution of spaces over B

An inverse system of the category Top_B is a colection $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathcal{A}\}$ of space X_α over B indexed by a directed set $\mathcal{A}$ and f.p. maps $p_{\alpha\alpha'} : X_{\alpha'} \longrightarrow X_\alpha$, $\alpha \le \alpha'$, such that $p_{\alpha\alpha'} \circ p_{\alpha'\alpha''} = p_{\alpha\alpha''}$ and $p_{\alpha\alpha} = 1_{X_\alpha}$, $\alpha \in \mathcal{A}$.

A morphism $f = \{f_\beta, \varphi\} : \underline{X} \longrightarrow \underline{Y} = \{Y_\beta, q_{\beta\beta'}, \mathcal{B}\}$ of inverse system of the category Top_B consists of a function $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ and of f.p. map $f_\beta : X_{\varphi(\beta)} \longrightarrow Y_\beta$, $\beta \in \mathcal{B}$, such that whenever $\beta \le \beta'$, then there is an index $\alpha \ge \varphi(\beta), \varphi(\beta')$ for which $f_\beta \circ p_{\varphi(\beta)} = q_{\beta\beta'} \circ f_{\beta'} \circ p_{\varphi(\beta')\alpha}$.

Two morphisms $\underline{f} = \{f_\beta, \varphi\}$, $\underline{g} = \{g_\beta, \psi\}: \underline{X} \longrightarrow \underline{Y}$ are said to be equivalent, $f \underset{B}{\simeq} g$, provided for each $\beta \in \mathcal{B}$ there is an $\alpha \in \mathcal{A}$, $\alpha \geq \varphi(\beta), \psi(\beta)$, such that $f_\beta \circ p_{\varphi(\beta)\alpha} = g_\beta \circ p_{\psi(\beta)\alpha}$.

Let pro-Top_B be a category, whose objects are the inverse systems $\underline{X}$ of the category Top_B and whose morphisms are the equivalence classes $[\underline{f}]_\beta$, relative to $\underset{B}{\simeq}$, of the morphisms $\underline{f}: \underline{X} \longrightarrow \underline{Y}$.

A morphism $\underline{p} = \{p_\alpha\}: X \longrightarrow \underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathcal{A}\}$ from a rudimentary system (X) to an inverse system $\underline{X}$ consists of the f.p. maps $p_\alpha: X \longrightarrow X_\alpha$, $\alpha \in \mathcal{A}$, such that $p_\alpha = p_{\alpha\alpha'} \circ p_{\alpha'}$, $\alpha \leq \alpha'$.

DEFINITION 1. *Let* X *be a space over* B *and let* $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathcal{A}\}$ *be an inverse system of the category* Top_B. *We say that* $\underline{p}: X \longrightarrow \underline{X}$ *is a resolution over B of the space* X *over B provided it satisfies the following two conditions:*

$R_B 1)$ *Let* $P \in ANR_B$, *let* $\mathcal{U}$ *be an open covering of* P *and let* $h: X \longrightarrow P$ *be a f.p. map. Then there exist an* $\alpha \in \mathcal{A}$ *and a f.p. map* $f: X_\alpha \longrightarrow P$ *such that* h *and* $f \circ p_\alpha$ *are* $\mathcal{U}$*-near.*

$R_B 2)$ *Let* $P \in ANR_B$ *and let* $\mathcal{U}$ *be an open covering of P. Then there is an open cover* $\mathcal{U}'$ *of* P *with the following property: if* $\alpha \in \mathcal{A}$ *and* $f, f': X_\alpha \longrightarrow P$ *are f.p. maps such that the f.p. maps* $f \circ p_\alpha$ *and* $f' \circ p_\alpha$ *are* $\mathcal{U}'$*-near, then there is an* $\alpha' \geq \alpha$ *such that the f.p. maps* $f \circ p_{\alpha\alpha'}$ *and* $f' \circ p_{\alpha\alpha'}$ *are* $\mathcal{U}$*-near.* ∎

If in a resolution over B, $p: X \longrightarrow \underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathcal{A}\}$ of the space X over B each X_α is an ANR_B, then we say that $\underline{P}$ is an ANR_B-resolution over B.

The next theorem is essential for construction of the fiber shape category.

THEOREM 1. *Every space* X *over a metrizable space* β *admits an* ANR_B*-resolution over B.*

In the proof of Theorem 1 we shall need the following lemma.

LEMMA 1. *Let* $f: X \longrightarrow Y$ *be a f.p. map from the topological space* X *over B to an* ANR_B*-space* Y *over B. Then there exists an* ANR_B*-space* Z *over B of weight*

$$w(Z) \leq \max\{w(X), w(B), \omega\}$$

and there exist f.p. maps $g: X \longrightarrow Z$ *and* $h: Z \longrightarrow Y$ *such that* $f = h \circ g$.

Proof. By Proposition 2., one can assume that f(X) is a closed subset of an ANR_B-space M, which satisfies $w(M) \leq \max(w(f(X)), w(B), \omega)$.

Since for metric spaces the weight coincides with the density we conclude that

$$w(f(X)) = s(f(X)) \leq s(X) \leq w(X)$$

Since Y is an ANR_B, there exists an open neighborhood Z of f(X) in M and there exists a f.p. map $h:Z\longrightarrow Y$, which extends the f.p. inclusion $i:f(X)\longrightarrow Y$. Let $g = i\circ f$. It is readily checked that $f = h\circ g$, $w(Z) \le w(M)$ and Z is an ANR_B-space over B. ■

Proof of Theorem 1. We say that two f.p. maps $p:X\longrightarrow P$, $p':X\longrightarrow P'$ are equivalent if there is a f.p. homeomorphism $h:P\longrightarrow P'$ such that $h\circ p = p'$. Let $\mathcal{A}$ consist of all equivalence classes of f.p. maps $p:X\longrightarrow P$, where P is an ANR_B-space over B of weight $w(P) \le \tau = \max\{w(X), w(\beta), \omega\}$.

For every α from the set $\mathcal{A}$ let $p_\alpha:X\longrightarrow Y_\alpha$ be a f.p. map from the class α. We order the set $\mathcal{B}$ of all finite subset $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of the set $\mathcal{A}$ by inclusion. The set $\mathcal{B}$ is a directed set. Let $Y_\beta = Y_{\alpha_1} \times Y_{\alpha_2} \times \dots \times Y_{\alpha_n}$ be the product of the space Y_{α_i} over B, $\alpha_i \in \beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, $i=1,n$, in the category Top_B, i.e. the pull-back of the maps $Y_\alpha\longrightarrow B$, $i = 1,2,\dots,n$.

For every $\beta \le \beta' = \{\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_m\}$ we define the f.p. map $q_{\beta\beta'}:Y_{\beta'}\longrightarrow Y_\beta$ as the projection

$$q_{\beta\beta'}(y_{\alpha_1}, \dots, y_{\alpha_n}, y_{\alpha_{n+1}}, \dots, y_{\alpha_m}) = (y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_n})$$

We next define the maps $q_\beta:X\longrightarrow Y_\beta$ as maps defined by the formula

$$q_\beta(x) = (q_{\alpha_1}(x), q_{\alpha_2}(x), \dots, q_{\alpha_n}(x)),\ x\in X$$

It is readily seen that Y_β is ANR_B-space over B and $q_{\beta\beta'}\circ q_{\beta'\beta''} = q_{\beta\beta''}$ for every $\beta \le \beta' \le \beta''$.

Note that $\underline{Y} = \{Y_\beta, q_{\beta\beta'}, \mathcal{B}\}$ is an ANR_B-system and $g=(q_\beta):X\longrightarrow \underline{Y}$ is a morphism of pro-Top_B.

Condition $R_B1)$ is an immediate consequence of Lemma 1.

We replace the inverse system $\underline{Y}$ by a larger inverse system. All pairs $\lambda = (\beta, \mathcal{U})$, where $\beta \in \mathcal{B}$ and $\mathcal{U}$ is an open neighborhood of $q_\beta(X)$ in Y_β, form a directed set Λ provided $\lambda = (\beta, \mathcal{U}) \le (\beta', \mathcal{U}') = \lambda'$ means that $\beta \le \beta'$ and $q_{\beta\beta'}(\mathcal{U}') \subseteq \mathcal{U}$.

For every $\lambda = (\beta, \mathcal{U}) \in \Lambda$ and $\lambda \le \lambda'$ we put $X_\lambda = \mathcal{U}$, $p_\lambda = q_\beta:X\longrightarrow \mathcal{U}$ and $p_{\lambda\lambda'} = q_{\beta\beta'}|_{\mathcal{U}'}:\mathcal{U}\longrightarrow \mathcal{U}'$. We obtain an ANR_B-inverse system $\underline{X}=\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$. It is readily checked that the morphism $\underline{p} = (p_\lambda):X\longrightarrow \underline{X}$ of category pro-Top_B satisfies conditions $R_B1)$ and $R_B2)$. ■

5. The fiber shape category Sh_B

In this section we give the construction of the fiber shape category. Our construction, given here, is based on the notion of resolution over B of topological spaces over a metrizable space B (comp.[MS], Ch.1).

DEFINITION 2. *Let* X *be a topological space over* B, $[\underline{X}] = \{X_\alpha, [p_{\alpha\alpha'}], \mathcal{A}\}$ *an inverse system in* $[Top_B]$ and $[P]: X \longrightarrow [\underline{X}]$ a *morphism of pro-*$[Top_B]$. *We call* $[\underline{p}]$ *an expansion over* B *of the space* X *over* B *provided it has the following properties:*

$E_B 1)$ *For every* ANR_B*-space* P *over* B *and f.p. map* $f: X \longrightarrow P$ *there is a* $\alpha \in \mathcal{A}$ *and a f.p. map* $h: X_\alpha \longrightarrow P$ *such that* $h \circ P_\alpha \underset{\beta}{\simeq} f$.

$E_B 2)$ *If* $f, f': X_\alpha \longrightarrow P$ *are f.p. maps,* $P \in ANR_B$ *and* $f \circ p_\beta \underset{B}{\simeq} f' \circ p_{\alpha'}$ *then there is a* $\alpha' \geq \alpha$ *such that* $f \circ p_{\alpha\alpha'} \underset{B}{\simeq} f' \circ p_{\alpha\alpha'}$. ∎

If all $X_\alpha \in ANR_B$, then $[\underline{p}]$ is called an ANR_B-expansion over B.

The main result of section 5 is the following theorem.

THEOREM 2. *Let* X *be a topological space over a metrizable space B. Then every resolution over B* $\underline{p}: X \longrightarrow \underline{X}$ *of* X *induces an* ANR_B *expansion over B of* X $[\underline{p}]: X \longrightarrow [\underline{X}]$

In the proof of Theorem 2 we need the following lemma.

LEMMA 2. *Let* X *be a topological space over the metrizable space B,* $P, P' \in ANR_B$, $f: X \longrightarrow P'$ *a f.p. map,* $h_0, h_1: P' \longrightarrow P$ *two f.p. maps such that* $h_0 \circ f \underset{B}{\simeq} h_1 \circ f$. *Then there exists an* ANR_B*-space* P" *over B and f.p. maps* $f': X \longrightarrow P"$, $h: P" \longrightarrow P'$ *such that* $h_0 \circ f' = f$ *and* $h_0 \circ h \underset{B}{\simeq} h_1 \circ h$.

Proof. Let $S: X \times I \longrightarrow P$ be a map such that $S(x,0) = (h_0 \circ f)(x)$, $S(x,1) = (h_1 \circ f)(x)$ and $\Pi_P \circ S = \Pi_{X \times I}$. Consider the subspace $C_B(I,P)$ of the space $C(I,P)$. Let $\Pi_{C_B(I,P)}: C_B(I,P) \longrightarrow B$ be the map given by

$$\Pi_{C_B(I,P)}(\varphi) = \Pi_P(\varphi(t))$$

Consequently, $C_B(I,P)$ is a space over B. The f.p. map $S: X \times I \longrightarrow P$ defines the map $s: X \longrightarrow C_B(I,P)$ such that $(s(x))(t) = S(x,t)$, $x \in X$, $t \in T$. The image of the point $x \in X$, $s(x) \in C_B(I,P)$, because $\Pi_P \circ s(x): I \longrightarrow B$ is a constant map. Indeed,

$$(\Pi_P \circ s(x))(t) = \Pi_P((s(x))(t) = \Pi_P(S(x,t)) = \Pi_{X \times I}(x,t) = \Pi_X(x)$$

for every $t \in I$.

For each $x \in X$ we have $(\Pi_{C_B(I,P)} \circ S)(x) = \Pi_{C_B(I,P)}(s(x)) = \Pi_P((s(x))(t) = \Pi_P(S(x,t)) = \Pi_{X \times I}(x,t) = \Pi_X(x)$. Thus, $\Pi_{C_B(I,P)} \circ s = \Pi_X$ and $s: X \longrightarrow C_B(I,P)$ is a f.p. map.

For all $x \in X$ we have $(s(x))(0) = S(x,0) = (h_0 \circ f)(x), (s(x))(1) = S(x,1) = (h_1 \circ f)(x)$.

Let $P' \times_B C_B(I,P) = \{(y,\varphi) \mid \Pi_{P'}(y) = \Pi_{C_B(I,P)}(\varphi)\}$. The map $f': X \longrightarrow P' \times_B C_B(I,P)$, given by $f'(x) = (f(x), s(x))$, is a f.p. map. Let

$$\Pi_{P' x_B C_B(I,P)}: P' \ x_B \ C_B(I,P) \longrightarrow B$$

be a map defined by

$$\Pi_{P' x_B C_B(I,P)}(y,\varphi)=\Pi_{P'}(y)=\Pi_{C_B(I,P)}(y).$$

Then we have

$$(\Pi_{P' x_B C_B(I,P)} \circ f')(x) = \Pi_{P' x_B C_B(I,P)}(f(x),s(x)) = \Pi_{P'}(f(x)) = \Pi_X(x).$$

Thus, $\Pi_X = \Pi_{P' x_B C_B(I,P)} \circ f'$.

It is clear that the first projection $h: P' x_B C_B(I,P) \longrightarrow P'$ is a f.p. map and $h \circ f' = f$.

We define the subset P" of $P' x_B C_B(I,P)$ to be the following:

$$P" = \{(y,\varphi) \in P' x_B C_B(I,P) \,|\, \varphi(0) = h_0(y),\ h_1(y) = \varphi(1)\}$$

The map $K: P' \ x_B \ C_B(I,P) \longrightarrow P$ is given by

$$K((y,\varphi),t) = \varphi(t),\ y \in P',\ \varphi \in C_B(I,P),\ t \in P$$

is a f.p. homotopy between $h_0 \circ h|_{P"}$ and $h_1 \circ h|_{P"}$.

Indeed, for every $(y,\varphi) \in P"$ and $t \in I$ we have

$$K(y,\varphi),0) = \varphi(0) = h_0(y) = h_0 \circ h(y,\varphi)$$

$$K(y,\varphi),1) = \varphi(1) = h_1(y) = h_1 \circ h(y,\varphi)$$

$$\Pi_{P' x_B C_B(I,P) x I}((y,\varphi),t) = \Pi_{P' x_B C_B(I,P)}(y,\varphi) = \Pi_{P'}(y) = \Pi_{C_B(I,P)}(\varphi) =$$

$$= \Pi_P(\varphi(t)) = \Pi_P(K(y,\varphi),t).$$

We shall prove that $P" \in ANR_B$. Now suppose that A is a closed subspace of a space Z over B and $l: A \longrightarrow P"$ is a map such that $\Pi_A = \Pi_{Z|A} \quad =_P \Pi \quad \circ l$.

Denote by $L: AxI \longrightarrow P$ the map defined by

$$L(a,t) = (h' \circ l(a))(t),\ (a,t) \in AxI$$

where h' is the second projection $P' x_B C_B(I,P) \longrightarrow C_B(I,P)$. L is a f.p. map, because

$$(\Pi_P \circ L)(a,t) = \Pi_P(L(a,t)) = \Pi_P((h' \circ l(a))(t) = \Pi_{C_B(I,P)}(h'(l(a))) =$$

$$= \Pi_A(a) = \Pi_{AxI}(a,t).$$

The map L is a f.p. homotopy between $h_0 \circ h \circ l$ and $h_1 \circ h \circ l$. Indeed,

$$L(a,0) = (h' \circ l(a))(0) = h_0 \circ h \circ l(a),$$

$$L(a,1) = (h' \circ l(a))(1) = h_1 \circ h \circ l(a),\ a \in A.$$

Observe that, since $P' \in ANR_B$ and $h \circ l: A \longrightarrow P'$ is a f.p. map, there is a neighborhood $\mathcal{U}$ of A in Z and there exists a f.p. map $\tilde{l}: \mathcal{U} \longrightarrow P'$ such that $\tilde{l}|_A = h \circ l$.

By Proposition 8 there exists a neighborhood $\mathcal{V}$ of A in $\mathcal{U}$ and a f.p. homotopy $\tilde{L}: \mathcal{V}xI \longrightarrow P$ between $h_0 \circ \tilde{l}'|\mathcal{V}$ and $h_1 \circ \tilde{l}'|\mathcal{V}$. Let $\tilde{l}"$ be a f.p. map $\tilde{l}": \mathcal{V} \longrightarrow C_B(i,P)$, given by $(\tilde{l}"(z))(t) = \tilde{L}(z,t)$, $z \in \mathcal{V}$, $t \in I$. For every $a \in A$ we have $(\tilde{l}''(a))(t) = \tilde{L}(a,t) = L(a,t) = (h'\circ l(a))(t)$. Consequently, $\tilde{l}|_A = h'\circ l$. We define the f.p. map $\tilde{l}:\mathcal{V}\longrightarrow P'\times_B C_B(I,P)$ by $\tilde{l}(z) = (\tilde{l}'(z),\tilde{l}''(z))$, $z \in \mathcal{V}$. For each $z \in \mathcal{V}$ we have

$$(\tilde{l}''(z))(0) = \tilde{L}(z,0) = h_o\circ\tilde{l}'(z)$$

$$(\tilde{l}''(z)(1) = \tilde{L}(z,1) = h_1\circ\tilde{l}'(z).$$

Consequently, $\tilde{l}:\mathcal{V}\longrightarrow P''$ is an extension of the f.p. map $l:A\longrightarrow P''$, which completes the proof of Lemma 2. ■

Proof of Theorem 2. The condition $E_B1)$ is an immediate consequence of Proposition 7. The proof of condition $E_B2)$ depends on Lemma 2. ■

Combining Theorem 1 with Theorem 2, one immediately obtains this result.

COROLLARY 1. *The homotopy category $[ANR_B]$ is a dense subcategory ([MS], Ch.1) of the homotopy category $[Top_B]$.* ■

The fiber shape category $\mathcal{S}h_B$ of topological space over a metrizable space B is, by definition, the abstract shape category $Sh_{(\mathcal{T},\mathcal{P})}$ ([MS], Ch. 1), where $\mathcal{T} = [Top_B]$, $\mathcal{P} = [ANR_B]$.

The object of the fiber shape category are all topological spaces over the metrizable space B. The morphism of $\mathcal{S}h_B$ from the space X over B to the space Y over B are given by triples $([p],[q],[f])$, where $[p]:X\longrightarrow[\underline{X}]$, $[q]:Y\longrightarrow[\underline{Y}]$ are ANR_B-expansion of X and Y respectively and $[f]:[\underline{X}]\longrightarrow[\underline{Y}]$ is a morphism of pro-$[Top_B]$. In order to define a fiber shape morphism $F:X\longrightarrow Y$, one chooses ANR_B-resolutions over B $\underline{p}:X\longrightarrow\underline{X}$ and $\underline{q}:Y\longrightarrow\underline{Y}$, which exist by Theorem1. and one chooses a morphism $[\underline{f}]:[\underline{X}]\longrightarrow[\underline{Y}]$ of pro-$[Top_B]$.

By Theorem 1 of ([MS], Sh.1.), for every morphism $[f]:X\longrightarrow Y$ of $[Top_B]$ and for ANR_B-expansions $[\underline{p}]:X\longrightarrow[\underline{X}]$, $[\underline{q}]:[\underline{X}]\longrightarrow[\underline{Y}]$ of pro-$[ANR_B]$ such that $[\underline{q}]\circ[\underline{f}] = [\underline{f}]\circ[\underline{p}]$. If we put $sh_B(X) = X$ and $sh_B([f]) = [\underline{f}]$ we obtain a covariant functor $sh_B:[Top_B]\longrightarrow Sh_B$. We call sh_B the fiber shape functor.

COROLLARY 2. *If $X \underset{B}{\simeq} Y$, then $sh_B(X) = sh_B(Y)$.* ■

REMARK. The results, obtained in this paper, are true also for category $Top^2(B,B_0)$ of closed pairs of topological spaces over a fixed pair (B,B_0) of metrizable spaces.

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FIBER TEORIJA OBLIKA I REZOLVENTE

U radu je definisana i proučena kategorija fiber oblika za sve topološke prostore nad datim metrizabilnim prostorom B. Definicija je zasnovana na prohomotopskoj kategoriji inverznih sistema apsolutnih fibrantskih okolinskih retrakta. Osnovnu ulogu u razmatranjima igraju rezolvente prostora nad B.

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