Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 5(1991), 87-96

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CONFIDENCE INTERVALS FOR THE MODE BASED ON ONE OBSERVATION

(Received 18.01.1991.)

Abstract. In this paper confidence intervals for the mode based on one observation are constructed. It is proved that there exist distributions for which the lenght of the confidence interval can be made in some sense arbitrary close to a non-attainable one. The method for constructing confidence intervals is generalized to multidimensional distributions.

1. Introduction

It is rather surprising that one can construct a confidence interval for the mean μ of a normal distribution $N(\mu,\sigma^2$ if there exists only one observation X from $N(\mu,\sigma^2)$ and nothing is known about the variance σ^2. Even more, there is a general method which allows to construct confidence intervals based on a single observation for the mode of an arbitrary unimodal distribution. A nice introduction can be found in [3]. According to the information given there, the main theorem was first proved in [1] and later generalized in [4] and [2]. At the beginning of our paper we introduce basic facts in a similar way as in [3]. Then we calculate confidence intervals for several frequently occurring families of distributions. The lengths of the confidence intervals depend on a positive constant k, which characterizes the given distribution or the whole class of distributions. We prove that there exist distributions such that they have an arbitrary small constant k. In some sense the length of the confidence interval approaches to a non-attainable lower limt in these cases. Then we modify the method in such a way that it gives confidence intervals for the mode of a multidimensional distribution when only one observation is available.

AMS Subject Classification (1991): 62F25

2. The Method

Throughout section 2-4 we assume that X is a continuous random variable and $t > 1$ is a given number.

LEMMA 2.1. *Let θ be a given real number. Let F be the distribution function of $X - \theta$. Define*

$$q = \begin{cases} F[-\theta t/(t+1)] - F[-\theta t/(t-1)] & \text{for} \quad \theta \geq 0 , \\ F[-\theta t/(t-1)] - F[-\theta t/(t+1)] & \text{for} \quad \theta < 0 . \end{cases}$$

Then we have

$$P(X-t|X| \leq \theta \leq X+t|X|) = 1-q .$$

Proof. If $\theta=0$ then the assertion obviously holds. Let $\theta \neq 0$. Then

$$P(X-t|X| \leq \theta \leq X+t|X|) = P(|X-\theta| \leq t|X|) = P(|1 - \frac{\theta}{X}| \leq t) =$$
$$= P(1-t \leq \frac{\theta}{X} \leq 1+t) .$$

Define

$$A = \{1-t \leq \frac{\theta}{X} \leq 1+t\} .$$

Then q is the probability that A does not occur. But a simple calculation gives that A does not occur if and only if X belongs to the interval with the end-point $\theta/(1-t)$ and $\theta/(1+t)$, i.e. if $X-\theta$ belongs to the interval with the end-points $-\theta t/(t-1)$ and $-\theta t/(t+1)$. If $\theta > 0$, then $-\theta t/(t+1)>-\theta t/(t-1)$, and if $\theta < 0$, then $-\theta t/(t-1)>-\theta t/(t+1)$. ∎

THEOREM 2.2. *Let X have a unimodal distribution with the mode θ. Then we have for arbitrary real a*

$$P(X-t|X-a| \leq \theta \leq X+t|X-a|) \geq 1 - \frac{2}{t+1} .$$

Proof. Let $a = 0$. Let F and f be the distribution function and the density of $X-\theta$, respectively. Since f is non-increasing on $[0,\infty)$, we have for $0 \leq s_1 < s_2$

$$(s_2-s_1)^{-1}\int_{s_1}^{s_2} f(s)ds \leq s_2^{-1}\int_0^{s_2} f(s)ds .$$

Thus

$$F(s_2)-F(s_1) \leq \frac{s_2-s_1}{s_2} [F(s_2)-F(0)] \leq \frac{s_2-s_1}{s_2} .$$

Similarly we have for $s_2 < s_1 \le 0$

$$F(s_1)-F(s_2) \le \frac{s_2-s_1}{s_2}[F(0)-F(s_2)] \le \frac{s_2-s_1}{s_2} \quad .$$

Using these inequalities we get for $\theta > 0$

$$F[-\theta t/(t+1)] - F[-\theta t/(t-1)] \le 2/(t+1),$$

and for $\theta < 0$

$$F[-\theta t/(t-1)] - F[-\theta t/(t+1)] \le 2/(t+1) \ .$$

The same inequality also holds for $\theta = 0$. Thus we proved

$$P(X-t|X| \le \theta \le X+t|X|) \ge 1 - \frac{2}{t+1} \ .$$

Now, we take $X - a$ and $\theta - a$ instead of X and θ, respectively. This yields

$$1 - \frac{2}{t+1} \le P(X-a-t|X-a| \le \theta-a \le X-a+t|X-a|) =$$
$$= P(X-t|X-a| \le \theta \le X+t|X-a|).$$

The proof is finished. ■

Theorem 2.2 says that for arbitrary a chosen before knowing X

$$(X-t|X-a|,\ X+t|X-a|)$$

is a confidence interval for θ on a level at least $1 - 2/(t+1)$. The choice of a can reflect our prior knowledge about the distribution of the random variable X.

THEOREM 2.3. *Let X have a unimodal distribution that is symmetric about its mode θ. Then for arbitrary fixed* a

$$P(X-t|X-a| \le \theta \le X+t|X-a|) \ge 1 - \frac{1}{t+1} \quad .$$

Proof. The proof is quite similar to that of Theorem 2.2. Since $X - \theta$ has a symmetric distribution about 0, we have

$$F(s_2)-F(0) \le \frac{1}{2} \quad \text{for } s_2 > 0$$

and

$$F(0)-F(s_2) \le \frac{1}{2} \text{ for } s_2 < 0. \blacksquare$$

THEOREM 2.4. *Let* X *have a unimodal distribution that is symmetric about its mode* θ. *Let* f *be the density of* X-θ. *The4n we have for arbitrary fixed* a

$$P(X-t|X-a| \le \theta \le X+t|X-a|) \ge 1 - \frac{2}{t-1} \sup_{c>0}[cf(c)].$$

Proof. It sufficies to consider the case a = 0. We use Lemma 1. First we show that

$$q = F[|\theta|t/(t-1)] - F[|\theta|t/(t+1)].$$

This obviously holds if $\theta < 0$. For $\theta \ge 0$ we use the relation $F(x) = 1-F(-x)$ since the distribution of $X - \theta$ is symmetric about 0. Thus

$$q \le \sup_{c>0}\{ F[c(t+1)/(t-1)] - F(c) \} =$$

$$= \sup_{c>0}\left[\int_c^{c(t+1)/(t-1)} f(x)dx \right] \le \sup_{c>0}\left[\left(c\,\frac{t+1}{t-1} - c\right)f(c) \right] =$$

$$= \frac{2}{t-1} \sup_{c>0}[cf(c)]$$

3. Special cases

In this section we apply Theorem 2.4 to some frequently used classes of distributions. The result presented in Theorem 3.1 is known (see [3]). We introduce it for sake of completness.

THEOREM 3.1. *Let* X *have a normal distribution with* EX=θ. *Then we have for arbitrary fixed* a

$$P(X-t|X-a| \le \theta \le X+t|X-a|) \ge 1 - \frac{1}{t-1}\sqrt{\frac{2}{\pi e}} = 1- \frac{0.484}{t-1}.$$

Proof. If $\mathrm{Var}X = \sigma^2$, then

$$\sup_{c>0}[cf(c)] = \sup_{c>0}\left[\frac{c}{\sigma\sqrt{2\pi}} \exp\left\{ - \frac{c^2}{2\sigma^2} \right\} \right] =$$

$$= \sup_{u>0}\left(\frac{u}{\sqrt{2\pi}} e^{-u^2/2} \right) = (2\pi e)^{-1/2}.$$

The assertion follows from Theorem 2.4. ■

THEOREM 3.2. *Let* X *have a logistic distribution with the distribution function*

$$G(x) = 1/[1+e^{b(\theta-x)}], \quad b > 0.$$

Then we have for arbitrary fixed a

$$P(X-t|X-a| \le \theta \le X+t|X-a|) \ge 1 - \frac{0.448}{t-1}.$$

Proof. The logistic distribution with the distribution function G is unimodal with the mode θ. The variable $X-\theta$ has the distribution function F and the density f, where

$$F(x) = 1/(1+e^{-bx}), \quad f(x) = be^{-bx}/(1+e^{-bx})^2.$$

We get

$$\sup_{c>0}[cf(c)] = \sup_{u>0}[\, ue^{-u}/(1+e^{-u})^2 \,].$$

The function $h(u)=ue^{-u}/(1+e^{-u})^2$ for $u \ge 0$ reaches its maximum in the point u_0 that is the root of the equation

$$1 - u + (1+u)e^{-u} = 0.$$

Numerical solution is $u_0 = 1.543\ 405$ and thus $h(u_0) = 0.223\ 872$. Now, we use again Theorem 2.4. ■

THEOREM 3.3. *Let* X *have a Laplace distribution with the density*

$$g(x) = \frac{1}{2b} \exp\left\{ -\frac{|x-\theta|}{b} \right\}, \quad b > 0.$$

Then we have for arbitrary fixed a

$$P(X-t|X-a| \le \theta \le X+t|X-a|) \ge 1 - \frac{1}{(t-1)e} = 1 - \frac{0.368}{t-1}.$$

Proof. In this case $X - \theta$ has the density

$$f(x) = (2b)^{-1} e^{-|x|/b}.$$

We get

$$\sup_{c>0}[cf(c)] = \sup_{c>0}\left[\frac{c}{2b} e^{-c/b}\right] = \sup_{u>0}\left[\frac{1}{2} ue^{-u}\right] = \frac{1}{2e}. \blacksquare$$

THEOREM 3.4. *Let X have a Cauchy distribution with the density*

$$g(x) = \frac{1}{\pi} \frac{b}{b^2+(x-\theta)^2}, \quad b > 0.$$

Then we have for arbitrary fixed a

$$P(X-t|X-a| \le \theta \le X+t|X-a|) \ge 1 - \frac{1}{(t-1)\pi} = 1 - \frac{0.318}{t-1}.$$

Proof. The density $f(x)$ of $X - \theta$ is $f(x) = \pi^{-1}b/(b^2+x^2)$. Then

$$\sup_{c>0}[cf(c)] = \sup_{c>0}\left[\frac{1}{\pi} \frac{bc}{b^2+c^2}\right] = \frac{1}{2\pi}. \blacksquare$$

4. Distributions minimizing a coefficient

For unimodal symmetric distribution we derived in Theorem 2.4 that $(X-t|X-a|,\ X+t|X-a|)$ is a confidence interval for the mode θ on a confidence level at least $1-(t-1)^{-1}k$, where

$$k = 2\sup_{c>0}[cf(c)],$$

where f is the density of $X - \theta$. If we want to ensure a confidence level at least $1 - \alpha$, we must choose $t = 1 + \frac{k}{\alpha}$. The lengh of such a confidence interval is

$$2\left(1 + \frac{k}{\alpha}\right)|X-a|.$$

We derive that k=0.484, 0.448, 0.368 and 0.318 for normal, logistic, Laplace and Cauchy distributions, respectively. From the definition it is clear that k>0. There is a question if for some distributions the constant k can be arbitrary small. The results of section 3 indicate that k is smaller for heavy-tailed distributions. This hint allows to prove that the answer to the question is affirmative.

THEOREM 4.1. *There exists a sequence of symmetric (about zero) unimodal densities* f_n *such that the corresponding constants*

$$k_n = 2\sup_{c>0}[cf(c)]$$

converge to 0 *as* $n \to \infty$.

Proof. Let $n \geq 2$. Define

$$f_n(x) = \begin{cases} 1/[(x+n)\ln(x+n)] & \text{for } 0 \leq x \leq n^{e^{0.5}} - n \quad , \\ 0 & \text{for } x > n^{e^{0.5}} - n \end{cases}$$

and

$$f_n(x) = f_n(-x) \quad \text{for } x < 0 .$$

It is clear that f_n is unimodal with the mode 0. Using the formula

$$\int \frac{dx}{(x+n)\ \ln(x+n)} = \ln[\ln(x+n)]$$

one can check that f_n is a density. But in our case

$$k_n = 2\sup_{c>0}[cf(c)] = 2\sup_{c>0}\left[\frac{c}{(c+n)\ \ln(c+n)}\right] \leq 2\sup_{c>0}\left[\frac{1}{\ln(c+n)}\right] = \frac{2}{\ln n}$$

and thus $k_n \to 0$. ∎

The example given in the proof of Theorem 4.1 can be modified in such a way that we get densities which are continuous and decreasing on $[0,\infty)$, but we omit details here.

5. Multidimensional distributions

The general approach can be modified also to multidimensional distributions. Our method is based on the Bonferroni inequality, which is frequently used in similar situations.

THEOREM 5.1. *Let a random vector* $X = (X_1,\ldots,X_n)'$ *have a continuous distribution such that the marginal density* $p_i(x)$ *is unimodal with a mode* θ_i, $(i = 1,\ldots,n)$. *Then for arbitrary fixed* $a_1,\ldots,a_n$ *and for every* $t_1 > 1, \ldots, t_n > 1$ *we have*

$$P(X_i - t_i|X_i - a_i| \le \theta_i \le X_i + t_i|X_i - a_i| \quad \text{for} \quad i = 1,\dots,n) \ge 1 - \sum_{i=1}^{n} \frac{2}{t_i + 1}\,.$$

If $p_i(x)$ *is symmetric about its mode* θ_i *for* $i = 1,\dots,n$, *then*

$$P(X_i - t_i|X_i - a_i| \le \theta_i \le X_i + t_i|X_i - a_i| \quad \text{for} \quad i=1,\dots,n) \ge 1 - \sum_{i=1}^{n} \frac{1}{t_i + 1}\,.$$

If the vector $X = (X_1,\dots,X_n)'$ *has a regular multidimensional normal distribution with* $EX=(\theta_1,\dots,\theta_n)'$, *then*

$$P(X_i - t_i|X_i - a_i| \le \theta_i \le X_i + t_i|X_i - a_i| \quad \text{for} \quad i=1,\dots,n) \ge$$

$$\ge 1 - \sqrt{\frac{2}{\pi e}} \sum_{i=1}^{n} \frac{1}{t_i - 1}\,.$$

Proof. Define the events

$$A_i = \{X_i - t_i|X_i - a_i| \le \theta_i \le X_i + t_i|X_i - a_i|\}$$

for $i = 1,\dots,n$. Let A_i^C be the complement of A_i. Then

$$P\left[\bigcap_{i=1}^{n} A_i\right] = 1 - P\left[\bigcup_{i=1}^{n} A_i^C\right] \ge 1 - \sum_{i=1}^{n} P(A_i^C).$$

But $P(A_i^C) \le 2/(t_i+1)$ for unimodal p_i (see Theorem 2.2), $P(A_i^C) \le 1/(t_i+1)$ for unimodal p that is symmetric about its mode θ_i (see Theorem 2.3) and $P(A_i^C) \le (2/\pi e)^{1/2}/(t_i-1)$ if p_i is a density of a normal distribution (see Theorem 3.1). This concludes the proof. ■

For example, if we want to construct a confidence interval for the mean $(\theta_1,\dots,\theta_n)'$ of an n-dimensional normal distribution on the confidence level at least $1-\alpha$, it suffices to find $t_1,\dots,t_n$ such that

$$\sqrt{\frac{2}{\pi e}} \sum_{i=1}^{n} \frac{1}{t_i - 1} = \alpha\,.$$

If we decide to take $t_1 = \dots = t_n = t$, then we get

$$t = 1 + \frac{n}{\alpha}\sqrt{\frac{2}{\pi e}}\,.$$

Similar calculations can be also made in other cases.

If a random vector $X = (X_1,\dots,X_n)'$ has a normal distribution with $EX = (\theta_1,\dots,\theta_n)'$, then this n-dimensional distribution is unimodal with the mode $(\theta_1,\ldots,\theta_n)'$ and, in the same time, the marginal distribution of X_i is unimodal with the mode θ_i, $i = 1,\ldots,n$. It must be stressed, however, that in general case the components of an n-dimensional mode may not be modes of the corresponding marginal one-dimensional distributions. We can demonstrate it on the following example.

Let Q be the square with the vertices $(0,0)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $(\sqrt{2},0)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Let $p(x_1,x_2) = 1$ for $(x_1,x_2) \in Q$ and $p(x_1,x_2) = 0$ otherwise. Then $(0,0)$ is a mode of this two-dimensional density, but the marginal distribution $p_1(x)$ has the mode $1/\sqrt{2}$. If this example is not convincing enough, it can be modified in the following way. Let $h(x_1,x_2)$ be the two-dimensional normal density with vanishing expectation and the unit variance matrix. Consider the density

$$p^*(x_1,x_2) = \varepsilon\ h(x_1,x_2) + (1-\varepsilon)\ p(x_1,x_2),$$

where $0< \varepsilon <1$. Then p^* is unimodal with the mode $(0,0)$ and for $0 \le \varepsilon < 2/[2+(2\pi e)^{-1}] = 0.892$ the one-dimensional marginal density p_1^* has still the mode $1/\sqrt{2}$. Really, the mode of p_1^* must be between 0 and $1/\sqrt{2}$. Since

$$p_1^*(x) = \varepsilon\ (2\pi)^{-1/2}\ e^{-x^2/2} + 2\ (1-\varepsilon)\ x\ , \quad \text{for} \quad 0\le x\le 1/\sqrt{2}$$

the derivative is

$$p_1^{*\prime}(x) = -\ \varepsilon\ x\ (2\pi)^{-1/2}\ e^{-x^2/2} + 2\ (1-\varepsilon).$$

We know already that $xe^{-x^2/2} \le e^{-1/2}$ and thus

$$p_1^{*\prime}(x) \ge -\ \varepsilon\ (2\pi e)^{-1/2} + 2\ (1-\varepsilon).$$

The right-hand side is positive if $\varepsilon < 2/[2+(2\pi e)^{-1/2}]$.

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INTERVALI POVERENJA ZA MOD DOBIJENI NA OSNOVU JEDNE OPSERVACIJE

U radu se konstruišu intervali poverenja za mod na osnovu samo jedne opservacije. Dokazuje se da postoje raspodele za koje se dužina intervala poverenja u nekom smislu može učiniti proizvoljno bliskom nedotiživoj. Metod za konstrukciju pomenutih intervala poverenja se uopštava i za slučaj višedimenzionih raspodela.

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