Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 5(1991), 81-86

Biljana Č. Popović

PREDICTIONS OF EAR(2) AND NEAR(2) TIME SERIES

(Received 17.12.1990.)

Abstract. Linear predictions of well known EAR(2) and NEAR(2) time series by method of Yaglom is presented.

Exponentialy distributed autoregressive time series of order two are treated here. Exactly, EAR(2) [1] and NEAR(2) [2] time series are treated. Known observations of the realization of these nonlinear models of time series generates linear Hilbert space by usual definitions of metric and norm. The possibility of predicting stationary time series uniquely by method of Yaglom [3], in such defined Hilbert space, includes the assumption that the spectral density $f(\tau)$ of the time series is a rational function by $e^{i\tau}$. Preciously, conditions

a) $g_s(z)$ is analytic function of complex variable z outside the unit circle and on it (on the circle $|z|=1$) - it may have singularities only inside the unit circle,

b) $g_s(\infty)=0$,

c) $[z^s-g_s(z)]f_1(z)$ is analytic function inside the circle $|z|=1$ and on it, where $f(\tau)=f_1(e^{i\tau})$;

are sufficient to define function $g_s(z)$ uniquely when $f_1(z)$ is rational. $g_s(z)$ will then be the spectral characteristic of the extrapolation.

Let us look at the definitions of the models EAR(2) and NEAR(2):

EAR(2)

$$(1)\quad X_t=\begin{cases} \alpha_1 X_{t-1}+\delta_t & \text{w.p. } 1-\alpha_2 \\ \alpha_2 X_{t-2}+\delta_t & \text{w.p. } \alpha_2 \end{cases}, \qquad 0<\alpha_1,\alpha_2<1 , \qquad t=0,\pm1,\pm2,\dots,$$

AMS Subject Classification (1991): 62M20

$\{\delta_t,\ t=0,\pm1,\pm2,\ldots\}$ is the sequence of independent identicaly distributed random variables:

$$
(2)\quad \delta_t=\begin{cases} 0 & \text{w.p.}\quad \alpha_1(1+\alpha_1-\alpha_2)^{-1} \\ E_t & \text{w.p.}\quad (1-\alpha_1)(1-\alpha_2)[1-(1+\alpha_1-\alpha_2)\alpha_2]^{-1} \\ (1+\alpha_1-\alpha_2)\alpha_2E_t & \text{w.p.}\quad (1-\alpha_2)(\alpha_1-\alpha_2)^2\{(1+\alpha_1-\alpha_2)[1-(1+\alpha_1-\alpha_2)\alpha_2]\}^{-1} \end{cases}
$$

where $\{E_t,\ t=0,\pm1,\pm2,\ldots\}$ is the sequence of independent identicaly exponentialy distributed with parameter μ, $\mu>0$, random variables.

NEAR(2)

$$
(3)\quad X_t=\begin{cases} \beta_1X_{t-1}+\delta_t & \text{w.p.}\quad \alpha_1 \\ \beta_2X_{t-2}+\delta_t & \text{w.p.}\quad \alpha_2 \\ \delta_t & \text{w.p.}\quad 1-\alpha_1-\alpha_2 \end{cases},\quad 0<\alpha_1,\alpha_2,\alpha_1+\alpha_2<1,\quad 0<\beta_1,\beta_2<1
$$

$t=0,\pm1,\pm2,\ldots,$

$\{\delta_t,\ t=0,\pm1,\pm2,\ldots\}$ is the sequence of independent identicaly distributed random variables:

$$
(4)\quad \delta_t=\begin{cases} E_t & \text{w.p.}\quad 1-p_2-p_3 \\ b_2E_t & \text{w.p.}\quad p_2 \\ b_3E_t & \text{w.p.}\quad p_3 \end{cases}
$$

$$p_2=[(\alpha_1\beta_1+\alpha_2\beta_2)b_2-(\alpha_1+\alpha_2)\beta_1\beta_2]/[(b_2-b_3)(1-b_2)]$$

$$p_3=[(\alpha_1+\alpha_2)\beta_1\beta_2-(\alpha_1\beta_1+\alpha_2\beta_2)b_3]/[(b_2-b_3)(1-b_3)]$$

$$0<b_3=\{(1-\alpha_1)\beta_1+(1-\alpha_2)\beta_2-[((1-\alpha_1)\beta_1+(1-\alpha_2)\beta_2)^2-4(1-\alpha_1-\alpha_2)\beta_1\beta_2]^{1/2}\}/2<$$

$$<b_2=\{(1-\alpha_1)\beta_1+(1-\alpha_2)\beta_2+[((1-\alpha_1)\beta_1+(1-\alpha_2)\beta_2)^2-4(1-\alpha_1-\alpha_2)\beta_1\beta_2]^{1/2}\}/2<1$$

We can recognize that their autocorrelations satisfy the difference equation of order two

$$(5)\quad \rho_r=A\rho_{r-1}+B\rho_{r-2}\quad,\quad r=1,2,\ldots$$

with the conditions

$$(6)\quad \rho_0=1\quad,\quad \rho_{-r}=\rho_r\quad.$$

The spectral densities of these two time series have the form

(7) $f(\tau)=C(2\pi\mu^2)^{-1}|e^{i\tau}-w_1|^{-2}|e^{i\tau}-w_2|^{-2}$

where w_1 and w_2 are the solutions of the equation

(8) $w^2-Aw-B=0$

and C is a real constant greater than zero.

Let's proove this statement.

If we use the above notations, the solution of difference equation (5) is given by the equation

(9) $$\rho_r = \frac{(A-w_1)[Aw_2+B(1-B)]}{(1-B)w_2^2(w_2-w_1)}\, w_2^r - \frac{(A-w_2)[Aw_1+B(1-B)]}{(1-B)w_1^2(w_2-w_1)}\, w_1^r$$

if $D=A^2+4B$ is greater than zero and under the assumption that $w_2>w_1$. The last assumption can be adopted without any loss of generality.

As

(10) $w_1+w_2=A$ and $w_1w_2=-B$,

it will be according to (9)

(11) $$\rho_r = \frac{1-w_1^2}{(1-w_1w_2)(w_2-w_1)}\, w_2^{r+1} - \frac{1-w_2^2}{(1+w_1w_2)(w_2-w_1)}\, w_1^{r+1} .$$

So, if

(12) $K_r=\mathrm{Cov}(X_t,X_{t+r})$, $r>0$

and as $K_r=K_{-r}$, we have

(13) $$K_h=\mu^{-2}\left[\frac{w_2(1-w_1^2)}{(1+w_1w_2)(w_2-w_1)}\, w_2^{|h|} - \frac{w_1(1-w_2^2)}{(1+w_1w_2)(w_2-w_1)}\, w_1^{|h|}\right] , \quad h=0,\pm1,\pm2,\ldots$$

Now the spectral density of those two time series has the form

(14) $$f(\tau) = (2\pi)^{-1}\sum_{h=-\infty}^{\infty} K_h e^{-i\tau h} =$$

$$= (2\pi\mu^2)^{-1}\,\frac{(1-w_1^2)(1-w_2^2)(1-w_1w_2)}{1+w_1w_2}\,|e^{i\tau}-w_1|^{-2}|e^{i\tau}-w_2|^{-2} .$$

The proof will be completed if we proove that C and D are greater than zero. This must be treated separately for each of those two time series.

For EAR(2) we have that

(15) $A=(1-\alpha_2)\alpha_1$, $B=\alpha_2^2$

and

(16) $D=(1-\alpha_2)^2\alpha_1^2+4\alpha_2^2$.

Obviously $D\geq 0$. D will be equal to zero if and only if $\alpha_1=\alpha_2=0$, but according to the definition of the model EAR(2), $0<\alpha_1,\alpha_2<1$, that is, $D>0$.

According to (10), w_1 and w_2 are such that one is greater and the other is less than zero and positive one has greater absolute value than negative one. As we have adopted that $w_2>w_1$, we can see that $|w_2|>|w_1|$ and their explicit forms are

(17) $w_1=[A-(A^2+4B)^{1/2}]/2$, $w_2=[A+(A^2+4B)^{1/2}]/2$

and it is easy to see that $|w_2|<1$. All this implies that $C>0$.

For NEAR(2) we have that

(18) $A=\alpha_1\beta_1$ and $B=\alpha_2\beta_2$

and

(19) $D = \alpha_1^2\beta_1^2+4\alpha_2\beta_2 > 0$

for all avaluable values of α_1, α_2, β_1, β_2 according to the definition of NEAR(2).

To proove that $C>0$, we must verify that $|w_1|<|w_2|<1$. The first inequality is trivial because of the same arguments as those in EAR(2) case. The second inequality is to be prooved with respect to β_1 and β_2, id est,

(i) $\beta_2\leq\beta_1<1$ gives $0\leq\beta_1-\beta_2<1-\beta_2$ and $\alpha_1\beta_1+\alpha_2\beta_2<1-\alpha_2(1-\beta_2)<1$,

(ii) $\beta_1\leq\beta_2<1$ gives $0\leq\beta_2-\beta_1<1-\beta_1$ and $\alpha_1\beta_1+\alpha_2\beta_2<1-\alpha_1(1-\beta_1)<1$.

These relations imply that $w_2<1$.

In the case of NEAR(2)

(20) $w_1=[\alpha_1\beta_1-(\alpha_1^2\beta_1^2+4\alpha_2\beta_2)^{1/2}]/2$, $w_2=[\alpha_1\beta_1+(\alpha_1^2\beta_1^2+4\alpha_2\beta_2)^{1/2}]/2$.

Now we can set the theorem:

THEOREM. *The best linear prediction for the value* X_{t+s} *(s is a nonnegative integer) from time series EAR(2) or NEAR(2) in Hilbert space* $H_{t-1}(X) = \mathrm{Clsp}\{X_{t-1}, X_{t-2}, \ldots\}$ *of known observations on the realization of the time series is*

$$(21)\quad \hat{X}_{t+s} = \frac{w_2^{s+2} - w_1^{s+2}}{w_2 - w_1} X_{t-1} - \frac{w_1 w_2 (w_2^{s+1} - w_1^{s+1})}{w_2 - w_1} X_{t-2} +$$

$$+\mu^{-1}\left[1 - \frac{w_2^{s+2} - w_1^{s+2} - w_1 w_2 (w_2^{s+1} - w_1^{s+1})}{w_2 - w_1} \right]$$

Proof. Let us define time series $\{Y_t,\ t=0,\pm1,\pm2,\ldots\}$ in the following way

$$(22)\quad Y_t = X_t - \mu^{-1}$$

for all t. Then we can assume that we have the observations $\{Y_{t-1}, Y_{t-2}, \ldots\}$ from the realization of the time series $\{Y_t\}$. This translation does not change the correlation structure and, according to that, the spectral density of the process. So,

$$(23)\quad f_1(z) = \frac{Cz^2}{(z-w_1)(1-w_1 z)(z-w_2)(1-w_2 z)}$$

and, with respect to *a)*, *b)* and *c)*, unique spectral characteristic of the proces $\{Y_t\}$ is

$$(24)\quad g_s(z) = (w_2 - w_1)^{-1}[(w_2^{s+2} - w_1^{s+2})z^{-1} - w_1 w_2 (w_2^{s+1} - w_1^{s+1})z^{-2}] \quad ,$$

id est,

$$(25)\quad g_s(e^{i\tau}) = (w_2 - w_1)^{-1}[(w_2^{s+2} - w_1^{s+2})e^{-i\tau} - w_1 w_2 (w_2^{s+1} - w_1^{s+1})e^{-2i\tau}]$$

and the best linear prediction in Hilbert space $H_{t-1}(Y)$ of the time series $\{Y_t\}$ for $s \geq 0$ periods ahead is

$$(26)\quad \hat{Y}_{t+s} = \frac{W_2^{s+2}-W_1^{s+2}}{W_2-W_1}\, Y_{t-1} - \frac{W_1 W_2 (W_2^{s+1}-W_1^{s+1})}{W_2-W_1}\, Y_{t-2}$$

and the result follows.

The error of such prediction will be the orthogonal distance between X_{t-1} and $H_{t-1}(X)$, $E(|X_{t+s}-\hat{X}_{t+s}|^2)$.

REFERENCES

[1] A.J. LAVRANCE & P.A.W. LEWIS, *The exponential autoregressive-moving average EARMA(p,q) process,* J.R. Statist. Soc. B 42, No.2 (1980), 150-161.

[2] A.J. LAWRANCE & P.A.W. LEWIS, *Modelling and residual analysis of nonlinear autoregressive time series in exponential variables,* J.R. Statist. Soc. B 47, No.2 (1985), 165-202.

[3] A.M. YAGLOM, *Stationary random functions,* Dover Publications, New York (1972).

Biljana Č. Popović

PROGNOZE VREMENSKIH SERIJA EAR(2) I NEAR(2)

Daje se linearna prognoza vremenskih serija EAR(2) i NEAR(2) metodom Jagloma.

Filozofski fakultet
Univerzitet u Nišu
18000 Niš
Yugoslavia

2. The Method

Throughout section 2-4 we assume that X is a continuous random variable and $t > 1$ is a given number.

LEMMA 2.1. *Let* θ *be a given real number. Let* F *be the distribution function of* $X - \theta$. *Define*

$$q = \begin{cases} F[-\theta t/(t+1)] - F[-\theta t/(t-1)] & \text{for} \quad \theta \ge 0 , \\ F[-\theta t/(t-1)] - F[-\theta t/(t+1)] & \text{for} \quad \theta < 0 . \end{cases}$$

Then we have

$$P(X-t|X| \le \theta \le X+t|X|) = 1-q .$$

Proof. If $\theta=0$ then the assertion obviously holds. Let $\theta\ne0$. Then

$$P(X-t|X| \le \theta \le X+t|X|) = P(|X-\theta| \le t|X|) = P(|1 - \frac{\theta}{X}| \le t) =$$
$$= P(1-t \le \frac{\theta}{X} \le 1+t) .$$

Define

$$A = \{1-t \le \frac{\theta}{X} \le 1+t\} .$$

Then q is the probability that A does not occur. But a simple calculation gives that A does not occur if and only if X belongs to the interval with the end-point $\theta/(1-t)$ and $\theta/(1+t)$, i.e. if $X-\theta$ belongs to the interval with the end-points $-\theta t/(t-1)$ and $-\theta t/(t+1)$. If $\theta > 0$, then $-\theta t/(t+1)>-\theta t/(t-1)$, and if $\theta < 0$, then $-\theta t/(t-1)>-\theta t/(t+1)$. ∎

THEOREM 2.2. *Let* X *have a unimodal distribution with the mode* θ. *Then we have for arbitrary real* a

$$P(X-t|X-a| \le \theta \le X+t|X-a|) \ge 1 - \frac{2}{t+1} .$$

Proof. Let $a = 0$. Let F and f be the distribution function and the density of $X-\theta$, respectively. Since f is non-increasing on $[0,\infty)$, we have for $0 \le s_1 < s_2$

$$(s_2-s_1)^{-1}\int_{s_1}^{s_2} f(s)ds \le s_2^{-1}\int_0^{s_2} f(s)ds .$$

Thus

$$F(s_2)-F(s_1) \le \frac{s_2-s_1}{s_2}\,[F(s_2)-F(0)] \le \frac{s_2-s_1}{s_2} .$$