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A NOTE ON STARLIKENESS OF CERTAIN INTEGRALS*

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Abstract. We consider the starlikeness of integral transform $F(z) = \frac{\alpha+1}{z^{\alpha}} \int_0^z t^{\alpha-1} f(t)dt$, where f is analytic in $|z|<1$, $f(0)=0$, and $\mathrm{Re}\{f'(z)\} > 0$.

1. Introduction and Preliminaries

Let A denote the class of function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.
By R we denote the class of functions $f \in A$ for which $\mathrm{Re}\{f'(z)\} > 0$, $z \in U$, and by S^* the class of starlike functions in U, i.e. the class of functions $f \in A$ such that $\mathrm{Re}\{zf'(z)/f(z)\} > 0$, $z \in U$. It is well-known that the classes R and S^* are the subclasses of univalent functions in U.

Let f and g be analytic in the unit disc U. The function f is subordinate to g, written $f \ll g$ or $f(z) \ll g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

In their paper [4], R. Singh and S. Singh have proved that if $f \in R$, then $F \in S^*$, where

$$F(z) = \int_0^z \frac{f(t)}{t}\, dt.$$

Later, Mocanu [3] has proved that the same is true for the transform

$$F(z) = \frac{2}{z} \int_0^z f(z)dt.$$

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In this note we show that the same result holds if F is defined by

$$F(z) = \frac{5/2}{z^{3/2}} \int_0^z t^{1/2} f(t)dt.$$

We note that we use a different approach as that one given in [3]. In that sense, we need the following lemmas.

Lemma 1. [2]. *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)),\ h(z) = \theta(q(z)) + Q(z)$$

and suppose that

(i) Q is starlike in U (univalent, but the condition $Q'(0) = 1$ is not necessary);

(ii) $Re\{ \frac{zh'(z)}{Q(z)} \} = Re\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \} > 0, \quad z \in U.$

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

(1) $$\theta(p(z)) + zp'(z)\phi(p(z)) \ll \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p \ll q$, and q is the best dominant of (1). ∎

We note that the univalent function q is said to be a dominant of (1) if $p \ll q$ for all p satisfying (1). If $\tilde{q}$ is a dominant of (1) and $\tilde{q} \ll q$ for all dominants q of (1), then $\tilde{q}$ is said to be the best dominant of (1). More facts about the method of differential subordination may be found in [2].

By using Lemma 1 we derive

Lemma 2. *If P is analytic in U with $P(0) = 1$ and if*

(2) $$Re\{zP'(z) + (\alpha+1)P(z)\} > 0, \quad z \in U,$$

for some α, $\alpha \geq 0$, then

(3) $$|\arg P(z)| < \gamma \frac{\pi}{2}, \quad z \in U,$$

where $0 < \gamma < 1$ is the root of the equation

(4) $$\gamma + \frac{2}{\pi} \operatorname{arctg} \frac{\gamma}{\alpha+1} = 1.$$

Proof. Let show that the following implication

(5) $$zP'(z) + (\alpha+1)P(z) \ll (\alpha+1) \frac{1+z}{1-z} \Rightarrow P(z) \ll \left(\frac{1+z}{1-z}\right)^{\gamma},$$

where $\alpha \geq 0$ and γ satisfies the equation (4), is true.

In Lemma 1 we choose $\theta(w) = (\alpha+1)w$, $\phi(w) = 1$ and $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$. Then

$$Q(z) = zq'(z)\phi(q(z)) = zq'(z) = \gamma \left(\frac{1+z}{1-z}\right)^{\gamma-1} \frac{2z}{(1-z)^2}$$

is starlike because $\underline{q}$ is a cinvex function in $\underline{U}$. The condition (ii) in Lemma 1 is also satisfied, and by applying (1) we have that if $\underline{p}$ is analytic in $\underline{U}$ with $p(0) = q(0) = 1$ and

$$(6)\qquad (\alpha+1)p(z) + zp'(z) \ll (\alpha+1)\left(\frac{1+z}{1-z}\right)^{\gamma} + \left(\frac{1+z}{1-z}\right)^{\gamma-1}\frac{2z}{(1-z)^2} = h(z)$$

then $p \ll q$ and $\underline{q}$ is the best dominant of (6).

Now let us show that

$$(7)\qquad (\alpha+1)\,\frac{1+z}{1-z} \ll h(z),$$

where $\underline{h}$ is defined in (6). Really, we have

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}\left[(\alpha+1) + \frac{2\gamma z}{1-z^2}\right] = q(z)h_1(z)$$

where we put

$$h_1(z) = \alpha + 1 + \frac{2\gamma z}{1-z^2}\,.$$

The function $\underline{q}$ maps the unit disc $\underline{U}$ onto the angle $|\arg w| < \gamma\frac{\pi}{2}$; the function h_1 maps $\underline{U}$ onto the complex plane minus the half-lines $\mathrm{Re}\{w\} = \alpha+1$, $\mathrm{Im}\{w\} \ge \gamma$ and $\mathrm{Re}\{w\} = \alpha+1$, $\mathrm{Im}\{w\} \le -\gamma$.

From where we easily obtain

$$|\arg h(e^{i\varphi})| \ge \gamma\,\frac{\pi}{2} + \mathrm{arctg}\,\frac{\gamma}{\alpha+1} = \frac{\pi}{2} \qquad (0 \le \varphi \le 2\pi)$$

which gives with $h(0) = \alpha+1$ that (7) is true. From (2) and (5) we get the statement of Lemma 2. ∎

Lemma 3. *If $\underline{P}$ is analytic in $\underline{U}$, with $P(0) = 1$ and*

$$(8)\qquad |\arg[zP'(z) + P(z)]| < \gamma\,\frac{\pi}{2}, \qquad z \in U,$$

for some γ, $0 < \gamma \le 1$, then

$$(9)\qquad |\arg P(z)| < \gamma_1\,\frac{\pi}{2},$$

where $0 < \gamma_1 < 1$ is the root of the equation

$$(10)\qquad \gamma_1 + \frac{2}{\pi}\,\mathrm{arctg}\,\gamma_1 = \gamma. \ ∎$$

The proof of this lemma is similar to the proof of Lemma 2.

Lemma 4. *Let $\underline{P}$ be a complex function such that*

$$(11)\qquad |\arg P(z)| \le \mathrm{arctg}\,\frac{\sqrt{3}}{\alpha}, \qquad z \in U,$$

for some $\alpha \ge 0$.

If $\underline{p}$ is analytic in $\underline{U}$, with $p(0) = 1$ and if

$$\mathrm{Re}\{P(z)[zp'(z) + p^2(z) + \alpha p(z)]\} > 0, \qquad z \in U,$$

then $\mathrm{Re}\{p(z)\} > 0, \quad z \in U.$ ∎

The proof is the same as in [3] for $\alpha = 1$.

2. Main result

THEOREM. *Let $f \in R$ and let*

$$F(z) = \frac{5/2}{z^{3/2}} \int_0^z t^{1/2} f(t)dt. \tag{12}$$

Then $F \in S^*$.

Proof. From (12) we obtain

$$zF'(z) + \frac{3}{2} F(z) = \frac{5}{2} f(z)$$

and

$$zF''(z) + \frac{5}{2} F'(z) = \frac{5}{2} f'(z). \tag{13}$$

Since $f \in R$, by applying Lemma 2 we get

$$|\arg F'(z)| < \gamma \frac{\pi}{2}, \quad z \in U, \tag{14}$$

where γ is the root of the equation

$$\gamma + \frac{2}{\pi} \operatorname{arctg} \frac{\gamma}{2.5} = 1, \qquad \gamma = 0.802\ldots \tag{15}$$

Further, let $p(z) = \dfrac{zF'(z)}{F(z)}$ and $P(z) = \dfrac{F(z)}{z}$.
Since

$$zP'(z) + P(z) = F'(z),$$

from Lemma 3 and (14) we have that

$$|\arg P(z)| < \gamma_1 \frac{\pi}{2},$$

where $\gamma_1 < 0.505\ldots$ is the root of the equation (10) for γ given in (15). Also, from (13) and since $\operatorname{Re}\{f'(z)\} > 0$, $z \in U$, we have

$$\operatorname{Re}\{P(z)[zp'(z) + p^2(z) + \frac{3}{2} p(z)]\} > 0, \quad z \in U.$$

Finally, because $|\arg P(z)| < \gamma_1 \frac{\pi}{2} < 0.793 \ldots < \operatorname{arctg} \frac{\sqrt{3}}{3/2} = 0.857 \ldots$ and from Lemma 4, we deduce that $\operatorname{Re}\{p(z)\} > 0$, $z \in U$, i.e. $F \in S^*$. ■

REMARKS. For $\beta \geq 0$, we denote by $R(\beta)$ the following set

$$R(\beta) = \{f \in A : \operatorname{Re}\{f'(z) + \beta z f''(z)\} > 0, \ z \in U\}.$$

By Lemma 2 we get $R(\beta) \subset R$. Also, we have that $R(\beta_1) \subset R(\beta_2)$ for $\beta_1 > \beta_2 \geq 0$. J. Krzyz [1] has shown that $R(0) = R \subset S^*$. The result of our theorem is equivalent to $R(2/5) \subset S^*$. Hence $R(\beta) \subset S^*$, for all $\beta \geq 2/5$, it means that this results improves the earlier results of R.Singh and S.Singh [4] and Mocanu [3]. In other words we have shown that integral transformation

$$F(z) = \frac{\alpha+1}{z^\alpha} \int_0^z t^{\alpha-1} f(t)dt$$

maps R into S^* for $0 \leq \alpha \leq \frac{3}{2}$.

The problem to find such a maximal α or an equivalent to find $\inf\{\beta : R(\beta) \subset S^*\}$ remains still open.

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BELEŠKA O ZVEZDOLIKOSTI IZVESNIH INTEGRALA

U radu je proučena zvezdolikost integralne transformacije $F(z) = \frac{\alpha+1}{z^{\alpha}} \int_0^z t^{\alpha-1} f(t)dt$, gde je f analitička funkcija u $|z| < 1$ koja zadovoljava uslove $f(0) = 0$ i $\mathrm{Re}\{f'(z)\} > 0$.

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