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ON α-CONVEXITY AND STARLIKENESS

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Abstract. In this paper we give a condition for α-convex function under which we have that $|\arg(zf'(z)/f(z))| < \gamma\pi/2$ $(0<\gamma\leq 1)$.

1. Introduction and preliminaries

First, we cite the following well-known definitions [4].

For a function $f(z)$ analytic in the unit disc $U = \{z : |z| < 1\}$, with $f'(0) \neq 0$ and $f(0) = 0$, we say that it is starlike if and only if $\mathrm{Re}\{zf'(z)/f(z)\} > 0$, $z \in U$.

For a function $f(z)$ analytic in U, with $f'(z) \neq 0$ we say that it is convex if and only if $\mathrm{Re}\{1+zf''(z)/f'(z)\} > 0$, $z \in U$.

Let A denote the class of functions $f(z)$ analytic in U with $f(0) = f'(0) - 1 = 0$, and let S^* and K denote the subclasses of A which consist of starlike and convex functions, respectively.

Further, let $J(\alpha, f(z))$ denote the operator

$$J(\alpha,f(z)) \equiv (1-\alpha)\,\frac{zf'(z)}{f(z)} + \alpha\,\left(1 + \frac{zf''(z)}{f(z)}\right), \tag{1}$$

where $\alpha > 0$ and $f(z) \in A$.

In [4] Mocanu has introduced the classes M_α of α-convex functions under the condition $\mathrm{Re}\{J(\alpha, f(z))\} > 0$, $z \in U$, $f(z) \in A$. It is well-known that

$$M_\alpha \subset S^*, \qquad \alpha > 0. \tag{2}$$

The relation (2) we may write in the form

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$$\mathrm{Re}\{J(\alpha,f(z))\} > 0 \quad \Rightarrow \quad \mathrm{Re}\{ \frac{zf'(z)}{f(z)} \} > 0, \quad z \in U, \quad \alpha > 0,$$

or equivalently:

(3) $$|\arg J(\alpha,f(z))| < \frac{\pi}{2} \quad \Rightarrow \quad |\arg \frac{zf'(z)}{f(z)}| < \frac{\pi}{2}, \quad z \in U, \quad \alpha > 0.$$

Now, we may put an expected question about constants β and γ, $0 < \beta, \gamma \leq 1$, such that the following implication

$$|\arg J(\alpha,f(z))| < \beta \frac{\pi}{2} \quad \Rightarrow \quad |\arg \frac{zf'(z)}{f(z)}| < \gamma \frac{\pi}{2} \quad (z \in U, \ \alpha > 0)$$

is true. In that sense, Theorem 1 in the next section gives an answer.

Also, we need some notions about subordination.

Let $f(z)$ and $F(z)$ be analytic in U. The function $f(z)$ is subordinate to $F(z)$, written $f(z) << F(z)$, if $f(z)$ is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$.

For the proof of our main result we use the following lemma due to Miller and Mocanu [2].

LEMMA A. *Let $q(z)$ be univalent in U and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) $Q(z)$ is starlike in U, and

(ii) $\mathrm{Re}\{ \frac{zh'(z)}{Q(z)} \} = \mathrm{Re}\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \} > 0, \quad z \in U.$

If $p(z)$ is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

(4) $$\theta(p(z)) + zp'(z)\phi(p(z)) << \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) << q(z)$, and $q(z)$ is the best dominant of (4). ∎

We note that the univalent function $q(z)$ is said to be a dominant of differential subordination (4) if $p(z) << q(z)$ for all $p(z)$ satisfying (4). If $\tilde{q}(z)$ is a dominant of (4) and $\tilde{q}(z) << q(z)$ for all dominants $q(z)$ of (4) then $\tilde{q}(z)$ is said to be the best dominant of (4).

More about differential subordinations we can find in [1] and [2].

2. Main result

THEOREM 1. *Let $J(\alpha,f(z))$ be defined by (1) for $\alpha > 0$ and $f(z) \in A$, and let $0 < \gamma \leq 1$ be given. If*

(5) $$|\arg J(\alpha,f(z))| < \beta \frac{\pi}{2}, \quad z \in U,$$

where

(6) $$\beta = \frac{2}{\pi} \operatorname{arctg} \left(\operatorname{tg} \frac{\gamma\pi}{2} + \frac{\alpha\gamma}{2\cos \frac{\gamma\pi}{2}} \right),$$

then

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \gamma \frac{\pi}{2} .$$

For the proof of this theorem we use the following

LEMMA 1. *Let $p(z)$ be analytic in U with $p(0) = 1$ and let $\alpha > 0$, $0 < \gamma \le 1$. If*

(7) $$p(z) + \alpha z \frac{p'(z)}{p(z)} \prec\prec \left(\frac{1+z}{1-z} \right)^{\beta}, \qquad z \in U,$$

where β is given by (6), then $p(z) \prec\prec \left(\frac{1+z}{1-z} \right)^{\gamma}$.

Proof. In Lemma A we choose $q(z) = \left(\frac{1+z}{1-z} \right)^{\gamma}$, $0<\gamma\le1$, and $\phi(w) = \frac{\alpha}{w}$, $\phi(w) = w$. Then we have that the function

$$Q(z) = zq'(z)\phi(q(z)) = \alpha \frac{zq'(z)}{q(z)} = \alpha\gamma \frac{2z}{1-z^2}$$

is really starlike in U, and

$$\operatorname{Re}\left\{ \frac{\theta'(q(z))}{\phi(q(z))} + z \frac{Q'(z)}{Q(z)} \right\} = \operatorname{Re}\left\{ \frac{1}{\alpha} q(z) + z \frac{Q'(z)}{Q(z)} \right\} > 0, \qquad z \in U,$$

i.e. the conditions (i) and (ii) of Lemma A are satisfied. By applying that lemma we get that if $p(z)$ is analytic in U with $p(0) = 1$, then the following implication

(8) $$p(z) + \alpha z \frac{p'(z)}{p(z)} \prec\prec q(z) + \alpha z \frac{q'(z)}{q(z)} = \left(\frac{1+z}{1-z} \right)^{\gamma} + \alpha \gamma \frac{2z}{1-z^2} \equiv h(z)$$

$$\longrightarrow \quad p(z) \prec\prec \left(\frac{1+z}{1-z} \right)^{\gamma}$$

is true and this is the best dominant.

For the function $h(z)$ defined in (8) we have

(9) $$h(e^{i\zeta}) = \left(i \operatorname{ctg} \frac{\zeta}{2}\right)^{\gamma} + \alpha\gamma \frac{i}{\sin \zeta} .$$

Since

$$i \operatorname{ctg} \frac{\zeta}{2} = \begin{cases} \operatorname{ctg} \frac{\zeta}{2}\, e^{i \frac{\pi}{2}} & , \quad 0 < \zeta < \pi \\ -\operatorname{ctg} \frac{\zeta}{2}\, e^{-i \frac{\pi}{2}} & , \quad -\pi < \zeta < 0, \end{cases}$$

from (9) we get

(10) $$h(e^{i\zeta}) = \left(\pm\operatorname{ctg} \frac{\zeta}{2}\right)^{\gamma}\cos\left(\pm\gamma \frac{\pi}{2}\right) + i \left[\left(\pm\operatorname{ctg} \frac{\zeta}{2}\right)^{\gamma}\sin\left(\pm\gamma \frac{\pi}{2}\right) + \frac{\alpha\gamma}{\sin \zeta}\right],$$

where we choose "+" in the case $0 < \zeta < \pi$ and "-" in the case $-\pi < \zeta < 0$. From (10) we easily conclude that $h(e^{i\zeta})$ is symetric on the real axis and we only consider the case $0 < \zeta < \pi$. In that case $\operatorname{Re}\{h(e^{i\zeta})\}$ and $\operatorname{Im}\{h(e^{i\zeta})\}$

are non-negative and if we put $\operatorname{ctg}\frac{\zeta}{2} = t$ $(0 < t < +\infty)$ we obtain

$$(11) \qquad \operatorname{tg}(\arg h(e^{i\zeta})) = \operatorname{tg}\frac{\gamma\pi}{2} + \frac{\alpha\gamma}{\cos\frac{\gamma\pi}{2}} \cdot \frac{1+t^2}{2t^{1+\gamma}} .$$

Further, for $0 < t \le 1$ we have

$$\frac{1+t^2}{2t^{1+\gamma}} \ge \frac{1+t^2}{2t} \ge 1,$$

while for $1 \le t \le +\infty$

$$\frac{2t^{1+\gamma}}{1+t^2} \le \frac{2t^2}{1+t^2} = 2 - \frac{2}{1+t^2} < 2, \qquad \text{i.e. } \frac{1+t^2}{2t^{1+\gamma}} > \frac{1}{2} .$$

In both these cases we have that $\frac{1+t^2}{2t^{1+\gamma}} > \frac{1}{2}$, and from (11) we get

$$\operatorname{tg}(\arg h(e^{i\zeta})) > \operatorname{tg}\frac{\gamma\pi}{2} + \frac{\alpha\gamma}{2\cos\frac{\gamma\pi}{2}} ,$$

which implies

$$\arg h(e^{i\zeta}) > \operatorname{arctg}\left(\operatorname{tg}\frac{\gamma\pi}{2} + \frac{\alpha\gamma}{2\cos\frac{\gamma\pi}{2}}\right) = \frac{\beta\pi}{2}, \qquad 0 < \zeta < \pi,$$

where β is defined by (6).

Therefore, for $0 < |\zeta| < \pi$ we have

$$(12) \qquad |\arg h(e^{i\zeta})| > \frac{\beta\pi}{2} .$$

From the previous inequality (12) we conclude that

$$(13) \qquad \left(\frac{1+z}{1-z}\right)^{\beta} \ll h(z).$$

Now, if the condition (7) is satisfied, then from (13) and (8) we finally get

$$p(z) \ll \left(\frac{1+z}{1-z}\right)^{\gamma}. \ \blacksquare$$

Proof of Theorem 1. If in Lemma 1 we put $p(z) = \frac{zf'(z)}{f(z)}$, where $f(z) \in A$, then we have the statement of theorem. ■

For $\gamma = 1$ in Theorem 1 we have $\beta = 1$ and the following

COROLLARY 1. *Let $f(z) \in A$ and $J(\alpha, f(z))$, $\alpha > 0$, be defined by (1). If*

$$|\arg J(\alpha, f(z))| < \frac{\pi}{2},$$

then

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2},$$

hich gives the relation (3). ■

If we put $\alpha = 1$ in Theorem 1, then we get

COROLLARY 2. *Let $f(z) \in A$ and $0 < \gamma \le 1$ be given. If*

$$\left|\arg \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \beta \frac{\pi}{2},$$

where

$$\beta = \frac{2}{\pi} \operatorname{arctg} \left(\operatorname{tg} \frac{\gamma\pi}{2} + \frac{\gamma}{2\cos \frac{\gamma\pi}{2}}\right),$$

then

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \gamma \frac{\pi}{2}. \blacksquare$$

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O α-KONVEKSNOSTI I ZVEZDOLIKOSTI

U radu se daje jedan uslov za α-konveksnost funkcija pri kome je ispunjeno $|\arg (zf'(z)/f(z))| < \gamma\pi/2$ $(0 < \gamma \le 1)$.

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