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FUNCTIONS WITH γ-CLOSED GRAPHS AND R-COMPACT SPACES

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Abstract. In this paper we introduce the notion of a γ-closed graph and obtain a characterization of R-compact spaces by utilizing functions with a γ-closed graph.

0. Introduction

In 1973, Kasahara [6] denoted by S a class of spaces containing the class of Hausdorff completely normal and fully normal spaces and utilized the class S to obtain a characterization of compact spaces. In 1975, Herrington and Long [5] obtained a characterization of H-closed spaces stated as follows:

THEOREM A (Herrigton & Long [5]). *A Hausdorff space Y is H-closed if and only if for every space X in the class S, each function $f:X \to Y$ with a strongly closed graph is weakly continuous.* ■

Quite recently, Bella and Cammaroto [1] have obtained the following characterization: a Urysohn space Y is Urysohn closed if and only if for every space $X \in S$, each function $f:X \to Y$ with a ν-closed graph is ν-continuous. Recently, in [3] and [4], the present authors have introduced and investigated the notions of almost γ-continuous functions and R-compact spaces, respectively. The purpose of this paper is to introduce functions with a γ-closed graph and to obtain a characterization of R-compact spaces parallel to Theorem A.

1. Preliminaries

Throughout the present paper spaces will always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let

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S be a subset and x a point of a topological space. The closure and the interior of S are denoted by $Cl(S)$ and $Int(S)$, respectively; the family $\mathcal{U}_x$ of all neighborhoods of x is called the neighborhood filter base of x. We denote by $\overline{\mathcal{U}}_x$ the closed filter on X having $\{Cl(V):V \in \mathcal{U}_x\}$ as a basis. Moreover, the neighborhood filter of $\overline{\mathcal{U}}_x$ will be denoted by $\mathcal{U}(\overline{\mathcal{U}}_x)$. A point $x \in X$ is said to be in the **γ-closure** of S, denoted by γ-$Cl(S)$, if $S \cap V \neq \emptyset$ for every $V \in \mathcal{U}(\overline{\mathcal{U}}_x)$. A subset S is said to be **γ-closed** if γ-$Cl(S) = S$. A subset S is said to be **regular-closed** if $Cl(IntS) = S$.

1.1. **DEFINITION.** *Let D be a directed set, $u:D \to X$ a net in a space X and $T_a = \{d \in D: a < d\}$. Then:*

(1) u is said to **γ-converge** *to $x \in X$ if for each $V \in \mathcal{U}(\overline{\mathcal{U}}_x)$ there is $a \in D$ such that $u(T_a) \subset V$;*

(2) u is said to **γ-adherent** *to $x \in X$ [1] if $u(T_a) \cap V \neq \emptyset$ for every a in D and every $V \in \mathcal{U}(\overline{\mathcal{U}}_x)$.* ■

1.2. **LEMMA.** *Let A be a subset and x a point of a space X. Then $x \in \gamma$-$Cl(A)$ if and only if there exists a net $u:D \to A$ $(\subset X)$ such that u γ-converges to x.* ■

1.3. **DEFINITION.** *Let S be a subset of a space X. (1) A cover $\{V_\alpha:\alpha \in \mathcal{A}\}$ of S by open sets of X is said to be* **regular** *if for each $\alpha \in \mathcal{A}$ there exists a nonempty regular closed set F_α of X such that $F_\alpha \subset V_\alpha$ and $S \subset \cup\{Int(F_\alpha): \alpha \in \mathcal{A}\}$; (2) S is said to be* **R-compact relative to X** *if every regular cover of S has a finite subcover; (3) A space X is said to be* **R-compact** *[4] if X is R-compact relative to X.* ■

1.4. **PROPOSITION.** *If K is a γ-closed subset of an R-compact space X, then K is R-compact relative to X.*

Proof. Let $\{V_\alpha:\alpha \in \mathcal{A}\}$ be any regular cover of K. Since K is γ-closed, for each $x \in X \setminus K$ there exists $V_x \in \mathcal{U}(\overline{\mathcal{U}}_x)$ and hence $W_x \in \mathcal{U}_x$ such that $x \in W_x \subset Cl(W_x) \subset V_x$ and $V_x \cap K = \emptyset$. Therefore, the family $\{V_\alpha:\alpha \in \mathcal{A}\} \cup \{V_x:x \in X \setminus K\}$ is a regular cover of X. Since X is R-compact, there exist a finite subset $\mathcal{A}_0$ of $\mathcal{A}$ and a finite number of points $x_1,\dots,x_n$ in $X \setminus K$ such that $X = [\cup\{V_\alpha:\alpha \in \mathcal{A}\}] \cup [\cup\{V_{x_i}:i = 1,2,\dots,n\}]$. Hence, we obtain $K \subset \cup\{V_\alpha:\alpha \in \mathcal{A}_0\}$. This shows that K is R-compact relative to X. ■

1.5. **DEFINITION.** *A function $f:X \to Y$ is said to be* **almost γ-continuous** *[3] if for each $x \in X$ and each $V \in \mathcal{U}(\overline{\mathcal{U}}_x)$ there exists $U \in \mathcal{U}_x$ with $f(U) \subset V$. A function $f:X \to Y$ is said to be* **weakly continuous** *[7] if for each $x \in X$ and each open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(U) \subset Cl(V)$.* ■

1.6. REMARK. It is shown in [3] that every weakly continuous function is almost γ-continuous but not conversely.

1.7. LEMMA. *For a function $f:X \to Y$ the following are equivalent:*

(a) f is almost γ-continuous.

(b) For each $x \in X$ and each net $u:D \to X$ converging to x, the net $f\cdot u: D \to Y$ γ-converges to $f(x)$.

(c) $f(Cl(A)) \subset \gamma\text{-}Cl(f(A))$ for every $A \subset X$.

Proof. (a) $\Rightarrow$ (b): Let $x \in X$ and let $u:D \to X$ be a net converging to x. Since f is almost γ-continuous, for each $V \in \mathcal{U}(\mathcal{U}_{f(x)})$ there exists $U \in \mathcal{U}_x$ such that $f(U) \subset V$. As u converges to x, there exists $a \in D$ such that $u(T_a) \subset U$. Therefore, we obtain $(f\cdot u)(T_a) \subset f(U) \subset V$ and hence the net $f\cdot u$ γ-converges to $f(x)$.

(b) $\Rightarrow$ (c): Let A be any subset of X and $x \in Cl(A)$. There exists a net $u:D \to A$ ($\subset X$) converging to x. Therefore, $f\cdot u:D \to f(A)$ ($\subset Y$) γ-converges to $f(x)$ and hence, by Lemma 1.2 $f(x) \in \gamma\text{-}Cl(f(A))$. Therefore, we have $f(Cl(A)) \subset \gamma\text{-}Cl(f(A))$.

(c) $\Leftrightarrow$ (a): This is proved in Theorem 2.2 of [3]. ∎

2. γ-closed graphs

For a function $f: X \to Y$, the subset $\{(x,f(x)): x \in X\}$ of the product space $X \times Y$ is called the **graph** of f and is denoted by $G(f)$. Long and Herrington [8] defined a function $f:X \to Y$ to have a **strongly closed graph** if for each $(x,y) \notin G(f)$ there exist open sets U and V containing x and y, respectively, such that $[U \times Cl(V)] \cap G(f) = \emptyset$.

2.1. DEFINICIJA. *For a function $f:X \to Y$ the graph $G(f)$ is said to be* **γ-closed with respect to Y** *(or briefly* **γ-closed***) if for every $(x,y) \notin G(f)$ there exist $U \in \mathcal{U}_x$ and $V \in \mathcal{U}(\bar{\mathcal{U}}_y)$ such that $[U \times V] \cap G(f) = \emptyset$.* ∎

2.2. LEMMA. *A function $f:X \to Y$ has a γ-closed graph if and only if for each $(x,y) \notin G(f)$ there exist $U \in \mathcal{U}_x$ and $V \in \mathcal{U}(\bar{\mathcal{U}}_y)$ with $f(U) \cap V = \emptyset$.* ∎

2.3. REMARK. It is obvious that any function with a γ-closed graph has also a strongly closed graph.

2.4. DEFINITION. *A space X is said to be* **strongly Urysohn** [2] *if for each pair of distinct points x,y in X there exist $U \in \mathcal{U}(\bar{\mathcal{U}}_x)$ and $V \in \mathcal{U}(\bar{\mathcal{U}}_y)$ such that $U \cap V = \emptyset$.* ∎

It is shown in [2;Example 1] that every strongly Urysohn space is Urysohn but not conversely.

2.5. THEOREM. *If $f:X \to Y$ is almost γ-contionuous and Y is strongly Urysohn, then G(f) is γ-closed.*

Proof. Let $(x,y) \notin G(f)$. Then $f(x) \neq y$ and hence there exist elements $W \in \mathcal{U}(\bar{\mathcal{U}}_{f(x)})$ such that $W \cap V = \emptyset$. Since f is almost γ-continuous, there is $U \in \mathcal{U}_x$ such that $f(U) \subset W$. Therefore, we have $f(U) \cap V = \emptyset$. It follows from Lemma 2.2 that G(f) is γ-closed. ∎

2.6. THEOREM. *If $f:X \to Y$ is weakly continuous and Y is Urysohn, then G(f) is γ-closed.*

Proof. Let $(x,y) \notin G(f)$. Then $f(x) \neq y$ and hence there exist open sets V and W such that $f(x) \in V$, $y \in W$ and $Cl(V) \cap Cl(W) = \emptyset$. Let $V_y = Y \setminus Cl(V)$. Then $y \in W \subset Cl(W) \subset V_y \in \mathcal{U}(\bar{\mathcal{U}}_y)$. Since f is weakly contionuous there exists $U_x \in \mathcal{U}_x$ such that $f(U_x) \subset Cl(V)$. Therefore, we obtain $f(U_x) \cap V_y = \emptyset$ and, by Lemma 2.2, G(f) is γ-closed. ∎

2.7. COROLLARY. (Long and Herrington [8]). *If $f:X \to Y$ is weakly continuous and Y is Urysohn, then G(f) is strongly closed.* ∎

2.8. THEOREM. *If a function $f:X \to Y$ has a γ-closed graph, then:*
(a) $f^{-1}(K)$ is closed in X for each K R-compact telative to Y;
(b) f(K) is γ-closed in Y for each compact set K in X.

Proof. We shall prove only the statement (a) since the statement (b) can be proved similarly. Let $x \notin f^{-1}(K)$. For each $y \in K$, $(x,y) \notin G(f)$ and as G(f) is γ-closed, by Lemma 2.2, there are $U_y \in \mathcal{U}_x$ and $V_y \in \bar{\mathcal{U}}_y$ such that $f(U_y) \cap V_y = \emptyset$. The family $\{V_y : y \in K\}$ is a regular cover of K. Since K is R-compact relative to Y, there exist finitely many points $y_1,\dots,y_n$ in K such that $K \subset \cup\{V_{y_i} : i = 1,\dots,n\}$. Let $U = \cap\{U_{y_i} : i = 1,\dots,n\}$. Then $U \in \mathcal{U}_x$ and $f(U) \cap K = \emptyset$. Therefore, we have $U \cap f^{-1}(K) = \emptyset$ and so $x \notin Cl(f^{-1}(K))$. This means that $f^{-1}(K)$ is closed in X. ∎

3. R-compact spaces

3.1. THEOREM. *Let Y be an R-compact space. For every space X each function $f:X \to Y$ with a γ-closed graph is almost γ-continuous.*

Proof. Let $x \in X$ and $V \in \mathcal{U}(\bar{\mathcal{U}}_{f(x)})$. There is $W \in \mathcal{U}_{f(x)}$ such that $f(x) \in W \subset Cl(W) \subset V$. For each $y \in Y \setminus W$, $(x,y) \notin G(f)$ and hence, by Lemma 2.2, there exist $U_{x,y} \in \mathcal{U}_x$ and $V_y \in \mathcal{U}(\bar{\mathcal{U}}_y)$ such that $f(U_{x,y}) \cap V_y = \emptyset$. The family $V \cup \{V_y : y \in Y \setminus W\}$ is a regular cover of Y. Since Y is R-compact there are finitely many points $y_1,\dots,y_n$ in $Y \setminus W$ such that $Y = V \cup \{V_{y_i} : i \leq n\}$. Put

$U = \cap\{U_{x,y_i} : i \le n\}$. Then we have $U \in \mathcal{U}_x$ and $f(U) \subset V$ which means that f is almost γ-continuous. ■

Following Kasahara [6], we denote by S a class of spaces containing the class of Hausdorff completely normal and fully normal spaces and use S to obtain a characterization of R-compact spaces.

3.2. **THEOREM.** *Let Y be a Urysohn space. If for every space $X \in S$, each function $f: X \to Y$ with a γ-closed graph is almost γ-continuous, then Y is R-compact.*

Proof. Assume that Y is not R-compact. We proceed to construct a space X in the class S and a function $f: X \to Y$ which has a γ-closed graph but is not almost γ-continuous. Since Y is not R-compact, there exists a net $u: D \to Y$ having no γ-adherent points in Y [4;Theorem 3.6]. Thus for every $y \in Y$ there exist $V_y \in \mathcal{U}(\bar{\mathcal{U}}_y)$ and $a \in D$ such that $u(T_a) \cap V_y = \emptyset$. Choose a point $\infty \notin D$ and put $X = D \cup \{\infty\}$. Following [6] we have a topological space X from S which has the property: if $U(\infty)$ is an open set containing ∞, then there exists $a \in D$ such that $T_a \cup \{\infty\} \subset U(\infty)$. Let $y^* \in Y$. Define a function $f: X \to Y$ by $f|D = u$ and $f(\infty) = y^*$. We shall prove that $G(f)$ is γ- closed. (i) Let $(x,y) \notin G(f)$ and $x \in D$. Since Y is Urysohn, there exist $A \in \mathcal{U}_{f(x)}$ and $B \in \mathcal{U}_y$ such that $Cl(A) \cap Cl(B) = \emptyset$. If $V = X \setminus Cl(A)$, then $y \in B \subset Cl(B) \subset V \in \mathcal{U}(\bar{\mathcal{U}}_y)$ and $f(x) \notin V$. Since $U = \{x\}$ is open in X, $f(U) \cap V = \emptyset$. (ii) Let $(x,y) \notin G(f)$ and $x = \infty$. Then $f(\infty) = y^* \ne y$ so that there exist $A \in \mathcal{U}_{y^*}$ and $B \in \mathcal{U}_y$ such that $Cl(A) \cap Cl(B) = \emptyset$. Put $V_1 = Y \setminus Cl(A)$. Then $y \in B \subset Cl(B) \subset V_1 \in \mathcal{U}(\bar{\mathcal{U}}_y)$ and $y^* = f(\infty) \notin V_1$. Since u has not γ-adherent points, $u(T_a) \cap V_y = \emptyset$ for some $a \in D$ and some $V_y \in \mathcal{U}(\bar{\mathcal{U}}_y)$. Therefore, for $V = V_1 \cap V_y \in \mathcal{U}(\bar{\mathcal{U}}_y)$ and $U = T_a \cup \{\infty\}$ we have $f(U) \cap V = \emptyset$. From (i) and (ii) it follows that $G(f)$ is γ-closed. Finally, we shall show that f is not almost γ-continuous. The injection $j: D \quad X$ defines a net in X converging to $\infty \in X$. However, $f \cdot j = u: D \to Y$ does not converge to $y^* = f(\infty)$ because y^* is not a γ-adherent point of u. It follows from Lemma 1.7 that f is not almost γ-continuous. ■

As an immediate consequence of Theorems 3.1 and 3.2 we have

3.3. **COROLLARY.** *A Urysohn space Y is R-compact if and only if for each space $X \in S$ every function $f: X \quad Y$ with a γ-closed graph is almost γ-continuous.* ■

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FUNKCIJE SA γ-ZATVORENIM GRAFOVIMA I R-KOMPAKTNI PROSTORI

U radu je uveden pojam funkcije sa γ-zatvorenim grafom i date neke osobine takvih funkcija. Jezikom γ-zatvorenog grafa daje se jedna karakterizacija R-kompaktnih prostora.

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