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ON NONASSOCIATIVE LEFT NEAR-RINGS WITH CERTAIN DESCENDING CHAIN CONDITION PROPERTY

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Abstract. Associative left or right near-rings S with the descending chain condition (dcc) on right or left (principal) S-subgroups have been considered by Laxton [1], Beidleman [2], Ligh [3] and others. We iniciate here a study of (nonassociative) left near-rings S with one or both dcc properties (*) and (**) . We obtain results on the existence of the identity of S, and the right or left invertibility of certain elements in S, particularly for the case when the additive group of S is simple.

1. Introduction

Let S be a left associative near-ring and x an element in S. The set $xS = \{xs | s \in S\}$ is a right S-subgroup and it is called the rigt principal or cyclic S-subgroup generated by x. In several investigations of associative near-rings S the descending chain condition (dcc) on right (principal) S-subgroups has been assumed ([1], [2], [3]). We consider here a (nonassociative) left near-ring S and we will frequently assume one (or both) descending chain conditions:

(*) For every x in S the descending chain $X_1 \supseteq X_2 \supseteq \ldots \supseteq X_n \supseteq \ldots$, where

(1) $$X_n = x(x(\ldots(xS)\ldots),\ x \text{ appearing n-times},$$

is stable i.e. for some n , $X_{n+k} = X_n$, for every $k \in N$.

(**) For every x in S the descending chain $X'_1 \supseteq X'_2 \supseteq \ldots \supseteq X'_n \supseteq \ldots$, where

(1') $$X'_n = (\ldots(Sx)\ldots x)x\ ,\ x \text{ appearing n-times}$$

is stable i.e. for some n , $X'_{n+k} = X'_n$, for every $k \in N$.

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2. Near-rings with the dcc property

For a (nonassociative) near-ring S, X_n does not have to be a right S-subgroup, but under certain associativity assumption X_n's will be S-subgroups.

LEMMA 1. *Let S be a (nonassociative) left near-ring and* $x \in S$. *All* X_n*'s are S-subgroups iff*

(2) $$(x, X_{n-1}, S) \subseteq X_n, \quad n \in N,$$

where $X_0 = S$ *and for the* $A, B, C \subseteq S$, $(A,B,C) = \{-a(bc)+(ab)c \mid a \in A, b \in B, c \in C\}$.

Proof. It is easy to see that every $X_n, n \in N$, is a subgroup of $(S,+)$. Supposing (2) we prove , by induction on n , that every X_n is a right S-subgroup. Let n = 1 and let $xs \in X_1, s_1 \in S$. Then:

$$(xs)s_1 = x(ss_1) - (-(xs)s_1 + x(ss_1)) \in X_1,$$

since $x(ss_1) \in X_1$ and by assumption $(x,S,S) \subseteq X_1$.

Let us suppose now that $n > 1$ and X_{n-1} is a right S-subgroup. Then for $x_n = x(x\ldots(xs)\ldots) \in X_n$ and $s_1 \in S$ we have

$$x_n s_1 = (xx_{n-1})s_1 = (xx_{n-1})s_1 - (-(xx_{n-1})s_1 + (xx_{n-1})s_1) \in X_n,$$

since $x_{n-1} \in X_{n-1}$ by inductive assumption, and $(x, X_{n-1}, S) \subseteq X_n$.

Conversely, let X_n be a right S-subgroup for $n \in N$. Then for n = 1 and $s, s_1 \in S$ we have $-x(ss_1) + (xs)s_1 \in X_1$, since $ss_1 \in S$ and $(xs)s_1 \in X_1$. For $n > 1$ and $s, s_1 \in S$ we have $-(xx_{n-1})s_1 + (xx_{n-1})s_1 = -(xx_{n-1})s_1 + x_n s_1 \in X_n$ since ,by assumption, $x_n s_1 \in X_n$ and $x_{n-1}s_1 \in X_{n-1}$, henceforth $(xx_{n-1})s_1 \in X_n$. ∎

REMARK 2. For an associative left near-ring S, $X_n = x^n S$, hence $x^n S \subseteq X_{n-1}$, $n \in N$. If S is nonassociative we define x^n recursively as follows:

$$x^1 = x \ , \text{ and } x^n = x * x^{n-1} \text{ for } n > 1.$$

Now we can prove the above inclusions under following assumption:

(3) $$(x, x^{n-1}, S) \subseteq X_{n-1} \text{ for } n > 1.$$

For n = 1 we have namely $x^n S = X_n \subseteq X_{n-1}$ without any assumptions. Let $n > 1$ and suppose that $x^{n-1}S \subseteq X_{n-2}$. Then for every $s \in S$

$$x^n s = x(x^{n-1}s) - (-(x^n s) + x(x^{n-1}s)) \in X_{n-1}.$$

We note that (3) follows from (2). For n = 2 we have by (2)

$$(x, x^{n-1}, S) \subseteq (x,S,S) \subseteq X_1 = X_{n-1},$$

and for $n > 2$

$$(x, x^{n-1}, S) \subseteq (x, X_{n-2}, S) \subseteq X_{n-1}.$$

Moreover if (2) is valid for n = 1 and for all $x \in S$, then $y^n S$ is a right S-subgroup for all $y \in S$ and for all $n \in N$. By Lemma 1, $y^n S$ is a right S-subgroup for some $y \in S$ and some $m \in N$ if and only if $(y^n, S, S) \subseteq y^n S$, and this is exactly (2) for n = 1 and $x = y^n$.

REMARK 3. By Lemma 1, every (nonassociative) near-ring S with dcc property on right S-subgroups satisfying condition (2), satisfies (*) also.

After the introductory comments on dcc (*) we start with:

THEOREM 4. *Let S be a (nonassociative) left near-ring satisfing (*). If there exists an element $a \in S$ which is not a left zero-divisor then a has a right identity $e = e_a$ and a right inverse a' ($aa' = e$). If moreover there exists at least one element $b \in S$ which is right regular ($xb = yb \Rightarrow x = y$) then a is the identityof S iff $(xe)y = x(ey)$ for all $x, y \in S$.*

Proof. Let a be a left regular element in S . Then by (*) we have $A_n = A_{n-1}$, $n \in N$ (for $x = a$ we write A_n instead of X_n). But $A_{n+1} = aA_{n-1}$, and since a is not a left zero-divisor a is left regular being left distributive. Consequently we can cancell a in the above equation as long till we get the equation for $n = 1$ i.e. $aS = a(aS)$. Canceling once more we get $S = aS$, hence a has a right identity $e = e_a$ and a right inverse a' in S.

Suppose now that b is a right regular element in S. If $e = e_a$ is the identity of S then $(xe)y = x(ey)$ for all $x, y \in S$. Conversely, let $(xe)y = x(ey)$ for all $x, y \in S$. Then for $x = a$ we have $ay = a(ey)$, hence $y = ey$ for all $y \in S$ (a is left regular). take now $y = b$. Now we have $(xe)b = xb$, hence $xe = x$ for all x because of the right regularity of b. ■

REMARK 5. If S is a (nonassociative) left near-ring satisfing $0x = x$ for all $x \in S$ then a right regular element $b \in S$ cannot be a right zero-divisor. Conversely, a right distributive element $b \in S$ which is not a right zero-divisor is right regular.

In the preceding proof it sufices that b is not a right zero-divisor if we assume $(xe-x)b = 0$ for all $x \in S$. If S is d.g. then $0x = 0$ for all $x \in S$. Moreover, in this case $(xe - x)b = 0$ for all $x \in S$ follows from the fact that $(xe)y = x(ey)$ for all $x, y \in S$. Really, let $b = \sum \pm s_i$ for some distributive elements $s_i \in S$. Then:

$$(xe-x)b = \sum \pm(xe-x)s_i = \sum \pm((xe)s_i - xs_i) = \sum \pm(xe)s_i - x(es_i)) = 0.$$

By Remark 5 we have the following corollary of Theorem 4:

THEOREM 6. *Let S be a d.g. (nonassociative) left near-ring satisfying (*). If there exists an element $a \in S$ which is not a left zero-divisor, then a has a right identity $e = e_a$ and a right inverse a' in S . Moreover, if there exists at least one element $b \in S$ which is not a right zero-divisor, then e is the identity of S iff $(xe)y = x(ey)$ for all $x, y \in S$.* ■

If S is associative, the condition $(xe)y = x(ey)$ for all $x, y \in S$ is automaticaly fulfilled. Moreover, the elements of S which are not zero-divisors form a multiplicative semigroup, hence all such elements have right (left) inverses iff they are invertible. From the two preceding theorems it follows:

COROLLARY 7. *Let S be an associative (d.g) left near-ring satisfing (*). S is unitary iff there exists at least one element $a \in S$ which is not a left zero-divisor and at least one element $b \in S$ which is right regular (not a right zero-divisor). In this case every $a \in S$ which is not a left zero-divisor is invertible, hence S is an associative division left near-ring iff S has no proper left zero-divisors (cf. Th.2. in [3]).* ■

3. Near-rings with dcc (*) and (**)

Similar, but not properly dual, results we can obtain using (**) instead of (*). The proofs are quite analogous and we formulate these results without proofs. Note that for any right distributive element x of a (nonassociative) left near-ring S, all X_n' defined in (1') are subgroups of $(S,+)$, and they are left S-subgroups iff

(2') $$(S, X_{n-1}', x) \subseteq X_n', \quad n \in N,$$

where $X_0' = S$.

For S an associative left near-ring $X_n' = Sx^n$, $n \in N$, hence $Sx^n \subseteq X_{n-1}'$, $n \in N$. If S is not associative and we define $x^n = x^{n-1}x$, for $n > 1$, the above inclusions remain true iff

(3') $$(S, x^{n-1}, x) \subseteq X_n', \quad n \geq 2.$$

Since $x^{n-1} \in X_{n-2}'$ for $n \geq 2$, (3') follows from (2'), for x a right distributive element of S. For such x, Sx^n is a subgroup of $(S,+)$ for all $n \in N$ and $Sx^n = Y_1'$ is a right S-subgroup if $(S,S,y) \subseteq Y_1'$ and $Y_1' = Sy$. Thus if (2') is valid for all right distributive $x \in S$, then for such $x \in S$ X_n' and Sx^n are right S-subgroups and $Sx^n \subseteq X_{n-1}'$ for all $n \in N$.

Consequently, in a (nonassociative) left near-ring S satisfying dcc on left S-subgroups, S satisfies condition (**) for all right distributive elements $x \in S$ if S satisfies condition (2') for such elements.

Now we go to the result we announced. The first one corresponds to Theorems 4 and 6.

THEOREM 8. *Let S be a (nonassociative) left near-ring satisfing (**). If $b \in S$ is a right regular element then b has a left unity $e = e_b$ and a left inverse b' in S . If moreover there exists at least one element $a \in S$*

which is not a left zero-divisor, then e is the identity of S iff $(xe)y = x(ey)$ for all $x,y \in S$. If S is distributive (that is to say if S is d.g. and S^2 additively commutative) then the above assertion holds for b which is not a right zero-divisor. ■

The folowing corollary of Theorem 8 is analogous to Corollary 7.

COROLLARY 9. *Let S be an associative (distributive) left near-ring satisfing (**). S is unitary iff there exists at least one element $a \in S$ which is not a left zero-divisor, and at least one element $b \in S$ which is right regular (not a right zero divisor). In this case an element $b \in S$ is right regular (not a right zero-divisor) iff b is invertible, hence S is an associative left division near-ring iff all nonzero elements of S are right regular.* ■

Now we combine the conditions (*) and (**). We consider first the non-associative case:

THEOREM 10. *Let S be a (nonassociative) left near-ring satisfing (*) and (**). Suppose that there exists an element $a \in S$ which is not a left zero-divisor and an element $b \in S$ which is right regular. Then a has a right identity e_a' and b has a left identity e_b'. Also S is unitary iff $(xe)y = x(ey)$ for $e = e_b'$ and $x,y \in S$. In this case every element in S which is not a left zero-divisor has a right inverse and every elment in S which is regular has a left inverse in S. If S is distributive (that is to say if S is d.g. and S^2 is additively commutative) then the above assertion holds for b not a right zero-divisor.* ■

From Theorem 10 it follows for the associative case:

COROLLARY 11. *Let S be an associative (distributive) near-ring which satisfies (*) and (**). S is unitary iff there exists $b \in S$ which is right regular (not a right zero-divisor). In this case all such elements are invertible.* ■

4. Near-rings with the simple additive group

We investigate here near-rings with the simple additive group starting with the following (well known) lemma (which we prove for the sake of completeness):

LEMMA 12. *Let S be a (nonassociative) left near-ring with the simple additive group. Suppose that there exists a nonzero right distributive element $s \in S$. Then either $ab = 0$ for all $a,b \in S$, or s is right regular.*

In the last case S has no nonzero left or right zero-divisors, hence every nonzero right distributive element in S is regular.

Proof. Define for $a \in S$, $T(a) = \{x \in S | ax = 0\}$. Since T(s) is a normal subgroup of the simple group (S,+), then $T(a) = \{0\}$ or $T(a) = S$. Since s is right distributive, $L(s) = \{y \in S | ys = 0\}$ is also a normal subgroup of (S,+), hence $L(s) = \{0\}$ or $L(s) = S$. If $L(s) = S$ then $as = 0$ for all $a \in S$, hence $T(a) = S$ i.e. $ab = 0$ for all $a,b \in S$. Let now $L(s) = \{0\}$. Then s is not a right zero-divisor, so s is right regular. But s is also left regular, since no nonzero element $a \in S$ can be a left and hence also a right zero divisor. Namely, such an element a is not in L(S), hence s is not in T(a) and thus $T(a) = \{0\}$. ■

Now we combine the above condition on the additive group of S with one or both of the conditions (*) and (**).

THEOREM 13. *Let S be a (nonassociative) left near-ring satisfing (*). Suppose that (S,+) is simple, and that there exists a nonzero right distributive element $s \in S$. Then either $ab = 0$ for $a,b \in S$ or s is right regular. If s is right regular, s has a right identity $e = e_s$ and S is unitary iff $(xe)y = x(ey)$ for all $x,y \in S$. In this case S has no nonzero left or right zero-divisors and every nonzero element in S has a right inverse in S.*

Proof. By Lemma 12 either $ab = 0$ for all $a,b \in S$, or s is right regular. In the last case S has no nonzero left or right zero-divisors and we can apply Theorem 4. ■

For S associative we have the following corollary of Theorem 13:

COROLLARY 14. *Let S be an associative left near-ring satisfing (*). Moreover suppose that the additive group (S,+) of S is simple and that there exists a nonzero right distributive element $s \in S$. Then S is unitary and every nonzero element in S is invertible or $ab = 0$ for all $a,b \in S$. If S is d.g. we can omit the assumption on right distributivity of s.* ■

Concerning the last assertion in the preceding corollary let us note something. If S is d.g. then L(s) is not necessarily a (normal) subgroup of (S,+) if s is not right distributive. But if s is different from zero and S is d.g., then $s = \sum \pm s_i$ for nonzero distributive s_i's. Hence we can take any s_i instead of s if s is not distributive.

The following two results are analogous to two preceding ones.

THEOREM 15. *Let S be a (nonassociative) left near-ring satisfing (**). Suppose that (S,+) is simple, and that there exists a nonzero right distributive element $s \in S$. Then either $ab = 0$ for $a,b \in S$ or s is right regular. If s is right regular, s has a left identity $e = e_s$ and S is unitary iff $(xe)y = x(ey)$ for all $x,y \in S$. In this case S has no nonzero left or right zero-divisors and every right distributive nonzero element in S has a left inverse in S.* ■

COROLLARY 16. *Let S be an associative left near-ring satisfing (**). Moreover suppose that the additive group (S,+) of S is simple and that there exists a nonzero right distributive element $s \in S$. Then $ab = 0$ for all $a,b \in S$ or s is right regular. In the last case S is unitary, S has no nonzero left or right zero-divisors and every nonzero right distributive element in S is invertible. (If S is d.g. we can omit the assumption on right distributivity of s but not also in the above assertion).* ■

Combining Theorems 13 and 14 we obtain the following result:

THEOREM 17. *Let S be an (nonassociative) left near-ring satisfying (**). Moreover suppose that the additive group (S,+) of S is simple and that there exists a nonzero right distributive element $s \in S$. Then either $ab = 0$ for all $a,b \in S$ or s is right regular. If s is right regular then s has a left identity $e' = e'_s$ and a right identity $e'' = e''_s$, and S is unitary iff $(xe)y = x(ey)$ for $e \in \{e',e''\}$, $x,y \in S$. In this case S has no left nor right proper zero-divisors, and each nonzero (right distributive) element in S has a right inverse (left inverse) in S. (Finally we note that for associative S Theorem* 17 *gives nothing more then Corollary* 14.*)* ■

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O NEASOCIJATIVNOM LEVOM SKORO PRSTENU SA USLOVIMA OPADAJUĆIH LANACA

Rezultati dobijeni u ovom radu odnose se na potrebne i dovoljne uslove egzistencije jediničnog elementa u (neasocijativnom) skoro-prstenu S i leve ili desne invertibilnosti nekih elemenata u S (specijalno kada je njegova aditivna grupa prosta).

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Correction to the paper:

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Page and Line	Set	For
6 16	$= x(x_{n-1}s_1) - \cdots + x(x_{n-1}s_1)$	$= (xx_{n-1})s_1 - \cdots (xx_{n-1})s_1$
6 17	since x_{n-1}	since $x_{n-1}s_1$
6 20	$-x(x_{n-1}s_1) + \cdots - x(x_{n-1}s_1)$	$-(xx_{n-1})s_1 + \cdots - (xx_{n-1})s_1$
6 21	$x(x_{n-1}s_1)$	$(xx_{n-1})s_1$
$6 - 4, -3$		
$-2, -1$	y^m	y^n
$6 \ -3$	all m	all n
7 8	identity of	identityof
7 19	Take	take
7 22	x for	0 for
$7 \ -7$	$((xe)s_1 - x(es_1)) = 0$	$((xe)s_1 - X(es_1)) = 0$
$8 \ -5$	results	result
9 18	e_a''	e_a'
9 19	some $e \in \{e_a'', e_b'\}$ and all	$e = e_a'$ and
9 20	element	elment
$9 \ -10$	exist an element $a \in S$ which is not a zero-divisor and an element	exists
10 3	$T(a)$	$T(s)$
10 10	$T(s)$	$T(S)$
10 14	satisfying	satisfing
11 16	a	an
11 17	$(*)$ and $(**)$	$(**)$
11 21	for some e	for e
	and all x, y	x, y