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ABOUT PARAMETRIC STOCHASTIC RESONANCE

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Abstract. Stability of trivial solution of a linear differential equations with the Markov coefficients is studied by the Lyapunov method. We construct asymptotic expansions for Lyapunov function of the linear equations with small parameter and apply this result to problem in the stability of the random oscillator.

1. Introduction

It is well known, that the equation of Mathé

$$\ddot{x} + \omega^2 (1 + h \cos \nu t) x = 0 \quad (1.1)$$

with the fraction $4\omega^2/\nu^2$ near to one, has two linear independent solutions, from which one increases exponential when $t \rightarrow \infty$ and the second decreases exponential (see, for example, [13]). More of that, it is easy to see that for small $\delta > 0$ and for $4\omega^2/\nu^2 \approx 1$, the equation

$$\ddot{x} + 2\delta \dot{x} + \omega^2 (1 + h \cos \nu t) x = 0 \quad (1.2)$$

has exponential increasing solution when $t \rightarrow \infty$, also. This phenomena is calling parametric resonance. Of course, when there exist a constant phase in parametric disturbance (i. e. when changing $\cos \nu t$ with $\cos(\nu t + \varphi)$ in (1.2)), the behaviour of the solutions is not changing.

Suppose now, that the phase of parametric disturbance $\varphi(t)$ is a stationary Markov process with the values in $[0, 2\pi]$, $E\{\varphi(t)\} = 0$, and consider the equation

$$\ddot{x} + 2\delta \dot{x} + \omega^2 (1 + h \cos(\nu t + \varphi(t))) x = 0 \quad (1.3)$$

In this case, when fraction of frequencies $4\omega^2/\nu^2 \approx 1$, it is

naturally to speak about stochastic parametric resonance. There are plenty papers about the behavior of the solutions of (1.3) (see lists in [10] and [14]). To the same problems are dedicating [3, 11, 12], in which is used a variation of the second method of Ljapunov, as suggested in [6] and developed for stochastic equations in [9]. In the following, we suggest a method of analyses exponential p -stability trivial solutions of (1.3), based on results in [11, 12].

2. Exponential stability in mean-square

Let us consider a differential equation in \mathbb{R}^n

$$\frac{dx}{dt} = A(y(t)) x \quad (2.1)$$

where $\{A(y), y \in \mathbb{Y}\}$ is a continuous matrix function on a compact \mathbb{Y} , $\{y(t), t \geq 0\}$ is a homogeneous stochastic continuous Feller-Markov process [1] on \mathbb{Y} with C -infinitesimal operator \mathcal{L} . We shall say that trivial solution of (2.1) is exponent p -stable for $p > 0$, if for some $c > 0$ and $\gamma > 0$, for every $y \in \mathbb{Y}$, $t \geq s \geq 0$ and $x \in \mathbb{R}^n$, for the solution of (2.1) with $x(s) = x$ when $y(s) = y$, is valid an inequality

$$\mathbb{E}|x(t)|^p \leq C e^{-\gamma(t-s)} |x|^2. \quad (2.2)$$

When $p=2$, we shall speak about exponent $**$ -stable in mean-square.

Let \mathbb{V} be a space of symmetric continuous matrix ($n \times n$) functions with the norm

$$v = \sup_{y \in \mathbb{Y}, |x|=1} |x^T v(y) x|,$$

$$\mathbb{K} = \{ v \in \mathbb{V}: x^T v(y) x \geq 0, \forall y \in \mathbb{Y}, \forall x \in \mathbb{R}^n \},$$

$$\mathring{\mathbb{K}} = \{ v \in \mathbb{V}: x^T v(y) x > 0, \forall y \in \mathbb{Y}, \forall x \neq 0 \}$$

The set \mathbb{K} is closed in \mathbb{V} , $\alpha_1 v_1 + \alpha_2 v_2 \in \mathbb{K}$ for every $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $v_1 \in \mathbb{K}$, $v_2 \in \mathbb{K}$ and from $v \in \mathbb{K}$, $-v \notin \mathbb{K}$ it follows $v \neq 0$. It is easy to see that from $v \in \mathring{\mathbb{K}}$, with $\varepsilon > 0$ small enough, it follows $\{g \in \mathbb{V}: g - v < \varepsilon\} \subset \mathbb{K}$, and beside that, the matrix unite I is in \mathbb{K} . The set \mathbb{K} , for which these conditions hold, is called a body conus [7], and the operators in $\mathbb{L}(\mathbb{V})$, which leaves \mathbb{K} invariant, are calling positive. Note that, in our case, $v \in \mathring{\mathbb{K}}$ if and only if there exist positive number c , such that for every $x \in \mathbb{R}^n$ and $y \in \mathbb{Y}$, $x^T v(y) x \geq c|x|^2$. Beside that, from the properties of symmetric matrix, it follows that for each $v \in \mathbb{V}$ there exist $v_1 \in \mathbb{K}$, $v_2 \in \mathbb{K}$ such that $v = v_1 - v_2$ and $v_1 \leq v$, $v_2 \leq v$ (reproduction of conus \mathbb{K} [7]).

Let \mathbb{B} be a space of symmetric matrix Baire functions on \mathbb{Y} with the norm $v = \sup_{y \in \mathbb{Y}, |x|=1} |x^T v(y) x|$. It is clear that $\forall v \in \mathbb{B}$. Let us define, when $t \geq 0$, operators in $\mathbb{L}(\mathbb{B})$ with the equality

$$(S^t v)(y) = \mathbb{E}_y \{ X^T(t) v(y(t)) X(t) \}, \quad (2.3)$$

where $X(t)$ is a matrix solution of (2.1), which satisfies a condition $X(0)=I$, and E_y is an conditional expectation when $y(0)=y$.

Theorem 2.1 The collection of operators $\{S^t, t \geq 0\}$ form a semigroup [4] on \mathbb{B} , which leaves space \mathbb{V} and cone \mathbb{K} invariant. The restriction of those operators on \mathbb{V} form a semigroup of class (C_0) with a generator

$$(\mathcal{A}v)(y) = A^T(y)v(y) + v(y)A(y) + (\mathcal{L}v)(y) \quad (2.4)$$

Proof. Let us denote with $P(t, y, x, \Gamma, G)$ the transition probability of Markov process $\{y(t), x(t), t \geq 0\}$. Let $x \in \mathbb{R}^n$, $y \in \mathbb{V}$, $v \in \mathbb{B}$. Then, for every $t \geq 0$, $\tau \geq 0$:

$$\begin{aligned} x^T (S^{t+\tau} v)(y) x &= E_y \{ x^T X^T(t+\tau) v(y(t+\tau)) X(t+\tau) x \} \\ &= \iint u^T v(z) u P(t+\tau, y, x, dz, du) \\ &= \iint u^T v(z) u \iint P(t, y, x, d\alpha, d\beta) P(\tau, \alpha, \beta, dz, du) \\ &= \iint P(t, y, x, d\alpha, d\beta) E_y \{ \beta^T X^T(\tau) v(y(\tau)) X(\tau) \beta \} \\ &= \iint P(t, y, x, d\alpha, d\beta) \beta^T S^\tau(v)(\alpha) \beta \\ &= x^T (S^t S^\tau v)(y) x \end{aligned}$$

From that it follows semigroup property of operators $\{S^t, t \geq 0\}$. Since matrix function $A(y)$ is continuous, the multiplication of elements $v \in \mathbb{V}$ on $A(y)$ defines continuous operator on \mathbb{V} . Beside that, from $v \in \mathcal{D}(\mathcal{L})$ (here, and in the following, the application of operator \mathcal{L} to the matrix function is on the elements) it follows that $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{L})$. So, the operator \mathcal{A} is a mapping from a dense subspace $\mathcal{D}(\mathcal{A})$ in \mathbb{V} to \mathbb{V} , i. e. the operator \mathcal{L} in the subspace $Cc\mathbb{V}$ has the same property [1]. It can be proved that for each real $\lambda > 2 \sup_{y \in \mathbb{V}} A(y) = a$ and each $g \in \mathbb{V}$ the equation $\mathcal{A}v - \lambda v = g$ has a solution $v \in \mathcal{D}(\mathcal{A}) \subset \mathbb{V}$ (i. e. the specter $\sigma(\mathcal{L})$ of the operator \mathcal{L} is in the half-plane $\{\lambda \in \mathbb{C}: \text{Re } \lambda \leq 0\}$ [1]), for which $\lambda v - \mathcal{A}v \geq (\lambda - a) v$. So, \mathcal{A} is a generator of some continuous semigroup $\{\tilde{S}^t, t \geq 0\}$ on \mathbb{V} [4]. For each $v \in \mathcal{D}(\mathcal{A}) \subset \mathbb{V}$ and each sufficiently small $t > 0$,

$$\begin{aligned} (S^t v)(y) &= E_y \left\{ \left(I + \int_0^t A^T(y(\tau)) d\tau \right) v(y(t)) \left(I + \int_0^t A(y(\tau)) d\tau \right) \right\} \\ &\quad + t(\mathcal{L}v)(y) + o(t), \end{aligned}$$

uniformly for $y \in \mathbb{V}$, from which follows that the generator of the restriction S^t on \mathbb{V} is \mathcal{A} , from (2.4), and that the semigroup \tilde{S}^t with the generator (2.4) is a restriction S^t on \mathbb{V} . The fact that S^t is invariant in \mathbb{K} follows immediately from the definition of \mathbb{K} . The theorem is proved. ■

Theorem 2.2. The trivial solution (2.1) is exponentially stable in mean-square if and only if there exist $r \in \mathbb{K}$ and $q \in \mathbb{K}$ such that

$$Aq = -r \quad (2.5)$$

Proof. Let $r \in \mathbb{K}$, $q \in \mathbb{K}$. Then, there exist such constants $c_1 > 0$ and $c_2 > 0$ such that for each $y \in \mathbb{V}$ and $x \in \mathbb{R}^n$ the inequality

$$c_1 |x|^2 \leq x^T r(y) x \leq c_2 |x|^2, \quad c_1 |x|^2 \leq x^T v(y) x \leq c_2 |x|^2. \quad (2.6)$$

is valid. Then

$$\begin{aligned} \mathbb{E}\{|x(t)|^2\} &\leq \frac{1}{c_1} \mathbb{E}\{x^T(t) q(y(t)) x(t)\} \\ &= \frac{1}{c_1} \mathbb{E}\{x^T X^T(t) q(y(t)) X(t) x\} = \frac{1}{c_1} x^T (S^t q)(y) x \end{aligned}$$

From the properties of continuous semigroup [4], for each $t \geq \tau \geq 0$ and $q \in \mathcal{D}(A)$ it follows an equation

$$S^t q = S^\tau q + \int_\tau^t S^u A q \, du,$$

from which, because of (2.4), (2.6) and (2.7),

$$\begin{aligned} x^T (S^t q)(y) x - x^T (S^\tau q)(y) x &= - \int x^T (S^u r)(y) x \, du \\ &\leq - \frac{c_1}{c_2} \int x^T (S^u r)(y) x \, du. \end{aligned}$$

Then

$$x^T (S^t q)(y) x \leq x^T q(y) x e^{-(c_1/c_2)t} \leq c_2 |x|^2 e^{-(c_1/c_2)t}$$

so the sufficiency follows from (2.7). To prove the necessary, we shall use

$$q(y) \stackrel{\text{def}}{=} \int_0^\infty \mathbb{E}_y \{ X^T(t) r(y(t)) X(t) \} dt = \int_0^\infty (S^t r)(y) dt \quad (2.8)$$

for arbitrary $r \in \mathbb{K}$. Then

$$\begin{aligned} (Aq)(y) &= \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \left\{ \int_0^\infty (S^t S^\tau r)(y) dt - \int_0^\infty (S^t r)(y) dt \right\} \\ &= \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \left\{ \int_0^\infty \mathbb{E}_y \{ X^T(t+\tau) r(y(t+\tau)) X(t+\tau) \} dt \right. \\ &\quad \left. - \int_0^\infty \mathbb{E}_y \{ X^T(t) r(y(t)) X(t) \} dt \right\} \\ &= -r(y). \end{aligned}$$

It is clear that for some $a > 0$, $c > 0$ and for each $t \geq 0$, from the continuousness of the matrix function $A(y)$ and from the compactness of \mathbb{V} it follows the inequality

$$X(t) \geq 1 - c \int_0^t e^{a\tau} X(\tau) d\tau$$

i. e. $x^T q(y)x > 0$ for each $x \neq 0$, $y \in V$, so from (2.8) and $r \in \mathbb{K}$ it follows $q \in K$. The theorem is proved. ■

The formula (2.8) determines the potential of the semigroup S^t on an element $r \in K^{**}$. Since the conus K is reproductive, the potential U is defined on V , i. e. from $K \subset \mathcal{D}(U)$ it follows $V \subset \mathcal{D}(U)$. So, the resolver R_λ of semigroup exist for each real $\lambda \geq 0$ if and only if $\mathcal{D}(U) \supset K$. But the operators S^t leave the reproductive conus K invariant, and by theorems about the specter of commutative set of operator which leaves the reproductive conus in Banach's space invariant [7], it can be proved that from $\mathcal{D}(U) \supset K$ follows $\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C}: \operatorname{Re} \lambda \leq -\rho\}$ for some $\rho > 0$. So, we have

Corollary 2.1. *The following statement are equivalent:*

- (a) *Trivial solution (2.1) is exponent stable in mean-square;*
- (b) *For each $r \in \mathbb{K}$ there exist*

$$q(y) \stackrel{\text{def}}{=} \int_0^\infty \mathbb{E}_y \{ X^T(t) r(y(t)) X(t) \} dt,$$

with $q \in \mathbb{K}$;

- (c) *There exist*

$$\tilde{q}(y) \stackrel{\text{def}}{=} \int_0^\infty \mathbb{E}_y \{ X^T(t) X(t) \} dt = \int_0^\infty (S^t I)(y) dt,$$

with $\tilde{q} \in \mathbb{K}$;

- (d) $\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C}: \operatorname{Re} \lambda \leq -\rho < 0\}$. ■

3. Stability in mean-square for small perturbations

Suppose that the differential equation (2.1) has coefficient near to constant, i. e.

$$\frac{dx}{dt} = \left[A_0 + \sum_{k=1}^m \varepsilon^k A_k(y(t)) \right] x, \quad (3.1)$$

where $\varepsilon \in [0, \varepsilon_0]$ and $\varepsilon_0 > 0$ are small enough, and the matrix functions $A_k(y)$ are continuous in y . Then, the operator \mathcal{A} from (2.4) can be rewrite as

$$\mathcal{A}(\varepsilon) = \sum_{k=1}^m \varepsilon^k A_k$$

where the operators A_k satisfy

$$(\mathcal{A}_0 v)(y) = A_0^T v(y) + v(y) A_0 + (\mathcal{L}v)(y),$$

$$(\mathcal{A}_k v)(y) = A_k^T(y) v(y) + v(y) A_k(y), \quad k=1, 2, \dots, m.$$

Lemma 3.1. *The specter $P_\sigma(\mathcal{A}_0)$ of the operator \mathcal{A}_0 can be set as*

$$P_\sigma(\mathcal{A}_0) = \{\lambda_1 + \lambda_2 + \lambda_3: \lambda_1, \lambda_2 \in \sigma(\mathcal{A}_0), \lambda_3 \in \sigma(\mathcal{L})\}. \quad (3.2)$$

A proof follows from a representation for V as tensor product

$$V = \mathbb{R}^n \otimes \mathbb{R}^n \otimes C(Y)$$

and the definition of the operator A_0 on a tensor $v(y) = (x \otimes u \otimes f)(y)$ $\stackrel{\text{def}}{=} xu^T f(y)$ by

$$(A_0 v)(y) = A_0^T x u^T f(y) + x u^T A_0 f(y) + x u^T (L f)(y).$$

Use the result of [15], and the lemma is proved.

Lemma 3.2. *Let the specter $\sigma(A_0)$ of the operator A_0 can be set as $\sigma(A_0) = \{0\} \cup \hat{\sigma}$, where the number zero is an isolation point of the specter with finite multiplicity 1. Then, there exist such positive number $\varepsilon_0 > 0$, so that for each $\varepsilon \in [0, \varepsilon_0]$ the space V can be disassembled in a sum of subspaces invariant relative to $A(\varepsilon)$*

$$V = V_0(\varepsilon) \oplus \overset{V}{V}_0(\varepsilon) \quad (3.3)$$

and the restriction $\Lambda(\varepsilon)$ of operator $A(\varepsilon)$ on $V_0(\varepsilon)$ has a discrete specter $\lambda_1(\varepsilon), \dots, \lambda_s(\varepsilon)$ with total multiplicity 1, $\lim_{\varepsilon \rightarrow 0} \lambda_j(\varepsilon) = 0$, and the restriction $\overset{V}{V}_0(\varepsilon)$ of operators $A(\varepsilon)$ on $\overset{V}{V}_0(\varepsilon)$ does not have the specter's point on a circle $\{z \in \mathbb{C}; |z| \leq r\}$ for some $r > 0$, and for each $\varepsilon \in [0, \varepsilon_0]$. ■

A proof follows from the facts that the operator A_0 is closed, the operators A_k , $k=1, 2, \dots, m$ are finite, and from the holomorphic properties of linear operators in Banach's space [5].

Corollary 3.1. *With the condition of Lemma 3.2 there is such base $F(\varepsilon) = \{f_1(\varepsilon), f_2(\varepsilon), \dots, f_l(\varepsilon)\} \subset V_0(\varepsilon)$, so*

$$A(\varepsilon) F(\varepsilon) = F(\varepsilon) \Lambda(\varepsilon), \quad (3.4)$$

where $\Lambda(\varepsilon)$ is a matrix $l \times l$, and $F(\varepsilon) \Lambda(\varepsilon)$ is a column-to-matrix product. The base $F(\varepsilon)$ and the matrix $\Lambda(\varepsilon)$ can be set as series in ε , $\varepsilon \in [0, \varepsilon_0]$:

$$F(\varepsilon) = F_0 + \varepsilon F_1 + \dots, \quad (3.5)$$

$$\Lambda(\varepsilon) = \Lambda_0 + \varepsilon \Lambda_1 + \dots, \quad (3.6)$$

where $\sigma(\Lambda_0) = \{0\}$, and F_0 is a base in V_0 .

Proof. Let F_0 be a base in V_0 and $P(\varepsilon)$ be a projector to the sum of root subspaces, corresponding to the points of specter $\lambda_1(\varepsilon), \dots, \lambda_s(\varepsilon)$ of the operator $A(\varepsilon)$. From [5] it follows that $P(\varepsilon)$ can be set as a series of degrees of ε on an interval $[0, \varepsilon_0]$, so, for small enough ε_0 , $F(\varepsilon) \stackrel{\text{def}}{=} P(\varepsilon) F_0$ is a base in $V(\varepsilon_0)$. Then use one of the methods for calculation the matrix of operator in finite dimensional space to prove the formulas (3.4), (3.5), (3.6). ■

Corollary 3.2. *With the conditions of Lemma 3.2 there is a positive number ε_0 , such that for each $\varepsilon \in (0, \varepsilon_0)$, exponentially stability of the trivial*

solution (3.1) is equivalent to the existence of a non-negative integer d , such that

$$\bar{q}^\varepsilon(y) = \sum_{k=-d}^{\infty} \varepsilon^k q_k(y), \quad (3.7)$$

$$A(\varepsilon) \bar{q}^\varepsilon(y) = -I, \quad (3.8)$$

$$\bar{q}^\varepsilon(y) \stackrel{\text{def}}{=} \sum_{k=-d}^{\infty} \varepsilon^k q_k(y) \in \mathbb{R}, \quad (3.9)$$

for each $\varepsilon \in (0, \varepsilon_0)$.

Proof. Let, for $\varepsilon \in (0, \varepsilon_0)$, the trivial solution (3.1) is exponentially stable in mean-square. Then $0 \notin \sigma(A(\varepsilon))$, for all $\varepsilon \in (0, \varepsilon_0)$, by the Corollary 2.1. Since the number zero is an isolating point in the specter of the operator A_0 of the finite multiplicity, the resolver $R_\lambda(\varepsilon)$ of the operator $A(\varepsilon)$ has, when $\lambda=0$, finite multiplicity pole [5], and, for small enough ε_0 , for each $\varepsilon \in (0, \varepsilon_0)$, $R_0(\varepsilon)$ can be stated as a series

$$R_0(\varepsilon) = \sum_{k=-d}^{\infty} \varepsilon^k C_k.$$

Then, we can find the solution of the equation (3.8) as a series (3.7). Since $\varepsilon \in (0, \varepsilon_0)$ is arbitrary, the matrix function $q_k(y)$, form (3.7), should satisfy conditions:

$$\begin{aligned} A_0 q_{-d} &= 0, \\ A_0 q_{-d+1} + A_1 q_{-d} &= 0, \\ \dots \\ A_0 q_{-1} + A_1 q_{-2} + \dots + A_m q_{-m-1} &= 0, \end{aligned} \quad (3.10)$$

$$A_0 q_0 + A_1 q_{-1} + \dots + A_m q_{-m} = -I, \quad (3.11)$$

$$A_0 q_k + A_1 q_{k-1} + \dots + A_m q_{k-m} = 0, \quad k \in \mathbb{N}. \quad (3.12)$$

When we find $q_{-d}, \dots, q_{-1}, q_0$ from (3.10)--(3.11), let us define a matrix function

$$\bar{p}^\varepsilon = \sum_{s=1}^m \varepsilon^{s-1} \sum_{j=1}^m A_j q_{-j+s}.$$

Now we have the solution (3.10)--(3.11), so we can write the matrix function \bar{q}^ε from (3.9). This function, by the construction, satisfies

$$A(\varepsilon) \bar{q}^\varepsilon = -I + \varepsilon \bar{p}^\varepsilon. \quad (3.13)$$

It is clear that for small enough ε_0 and for each $\varepsilon \in (0, \varepsilon_0)$

$$-I + \varepsilon \bar{p}^\varepsilon \in \mathbb{R}$$

and by the condition of the Corollary 2.1 $0 \notin \sigma(A(\varepsilon))$, i. e. the solution

(3.13) \hat{q}^ε is unique, and, again by the Corollary 2.1, $\hat{q}^\varepsilon \in \mathbb{K}$ for each $\varepsilon \in (0, \varepsilon_0)$.

Now let the solution (3.7) of the equation (3.8) exist and let (3.9) is valid for each $\varepsilon \in (0, \varepsilon_0)$. Then, for small enough ε_0 and for each $\varepsilon \in (0, \varepsilon_0)$, q^ε should be in \mathbb{K} , by (3.7), and then, by the Corollary 2.1, the trivial solution (3.1) is exponentially stable in mean-square. The corollary is proved. ■

Note that

the number d is not fixed, but defined from the condition that (3.11) is valid. As an example, consider an equation in \mathbb{R}^n

$$\frac{dx}{dt} = \varepsilon A(y(t)) x \quad (3.14)$$

where $\{y(t), t \geq 0\}$ is a

homogeneous Feller-Markov process on a compact \mathbb{V} with an infinitesimal operator \mathcal{L} .

Corollary 3.3. Let $\sigma(\mathcal{L}) = \{0\} \cup \hat{\sigma}$, where $\hat{\sigma} = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -\rho < 0\}$ and the multiplicity of zero is equal to one (uniform ergodicity). If μ is an invariant measure of Markov process $\{y(t), t \geq 0\}$,

$$\bar{A} = \int_{\mathbb{V}} A(y) \mu(dy), \quad \sigma(\bar{A}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\},$$

then the trivial solution (3.14) is exponentially stable in mean-square for each $\varepsilon \in (0, \varepsilon_0)$ and for small enough $\varepsilon_0 > 0$.

Proof. Because of all conditions, there exist [13] a positive definite matrix \bar{q} , which satisfies the equation of Lyapunov

$$\bar{A}^T \bar{q} + \bar{q} \bar{A} = -I.$$

Let $q_0 \in \mathbb{V}$ and

$$q^\varepsilon = \bar{q}/\varepsilon + q_0.$$

Then

$$\begin{aligned} \mathcal{A}(\varepsilon) q^\varepsilon(y) &= A^T(y) \bar{q} + \bar{q} A(y) + \mathcal{L} q + \frac{1}{\varepsilon} (\mathcal{L} \bar{q})(y) \\ &\quad + (\mathcal{L} q_0)(y) + \varepsilon [A^T(y) q_0(y) + q_0(y) A(y)]. \end{aligned}$$

Since $\mathcal{L} \bar{q} = 0$, the equation (3.10) is solved. The equation in (3.11) has the form

$$(\mathcal{L} q_0)(y) = -(A^T(y) \bar{q} + \bar{q} A(y)) - I. \quad (3.16)$$

This equation has a solution if and only if the right-hand side of (3.16) is orthogonal to the kernel of conjugate operator. Since all elements in the kernel $N(\mathcal{A}_0^*)$ have the tensor form $p \otimes \mu$, where p is any symmetric $n \times n$ matrix (note that $\mathbb{V} = \mathbb{R}^n \otimes \mathbb{R}^n \otimes C(\mathbb{V})$, i. e. $\mathbb{V} = \mathbb{R}^n \otimes \mathbb{R}^n \otimes C^*(\mathbb{V})$) and, because of the conditions of Corollary, the equation (3.15) has a solution \bar{q} , then, from the definition of scalar product and from the equation (3.15) we have

$$\int_{\mathbb{V}} \operatorname{Sp}[(A^T(y) \bar{q} + \bar{q} A(y)) p + p] \mu(dy) = \operatorname{Sp}[(\bar{A}^T \bar{q} + \bar{q} \bar{A} + I) p] = 0$$

for any matrix p . Then, the equation (3.16) has a solution, so the equation

$$A(\varepsilon)q^\varepsilon = -I + \varepsilon\tilde{q},$$

where $\tilde{q}(y) = A^T(y)q_0(y) + q_0(y)A(y)$, has a solution in \mathbb{K} for small enough ε_0 and each $\varepsilon \in (0, \varepsilon_0)$. From $I - \varepsilon\tilde{q} \in \mathbb{K}$, for each $\varepsilon \in (0, \varepsilon_0)$ and small enough $\varepsilon_0 > 0$, and from the Corollary 2.1 we see that the proved statement is valid. The analogue result, which uses the another method, was present in [8]. ■

4. Stochastic oscillator.

In the beginning we investigate the stability of a simple oscillator, defined by a deterministic equation of second order

$$\ddot{x} + 2\gamma\varepsilon^2\dot{x} + (1 + \varepsilon\Delta + \varepsilon h \cos 2t) x = 0 \quad (4.1)$$

The small parameter ε simplifies the analyse of the stability. If we use the vector forms, then, by the method of the last section

$$\frac{d\vec{x}}{dt} = (A_0 + \varepsilon A_1(y(t)) + \varepsilon^2 A_2) \vec{x} \quad (4.2)$$

Here $\{y(t)\}$ is a "Markov" process on $[0, \pi]$ with a C -infinitesimal operator $(\mathcal{L}v)(y) = dv(y)/dy$ in the space of continuous functions which satisfy the condition $v(0) = v(\pi)$. The matrix A_0 , A_1 and A_2 are

$$A_1(y) = -h \cos 2y \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \Delta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = -\begin{pmatrix} 0 & 0 \\ 0 & 2\gamma \end{pmatrix}.$$

The operators \mathcal{A}_0 , \mathcal{A}_1 , \mathcal{A}_2 in the space \mathbb{V} of symmetric matrix functions, which satisfy the condition $v(0) = v(\pi)$, have the forms

$$\begin{aligned} (\mathcal{A}_0 q)(y) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} q(y) + q(y) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \frac{dq(y)}{dy}, \\ (\mathcal{A}_1 q)(y) &= (-h \cos 2y - \Delta) \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} q(y) + q(y) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right], \\ (\mathcal{A}_2 q)(y) &= - \begin{pmatrix} 0 & 0 \\ 0 & 2\gamma \end{pmatrix} q(y) - q(y) \begin{pmatrix} 0 & 0 \\ 0 & 2\gamma \end{pmatrix}. \end{aligned}$$

If we define a scalar product in \mathbb{V} with

$$\langle p, q \rangle = - \int \text{Sp } p(y)q(y) dy,$$

then

$$(\mathcal{A}_0 p)(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p(y) + p(y) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{dp(y)}{dy}.$$

It is easy to check that the number zero is an isolating point of the specter of \mathcal{A}_0 with the multiplicity 3, and a base is of the form $F_0 = \{f_1, f_2, f_3\}$, where $f_1 = I$,

$$f_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cos 2y + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sin 2y, \quad f_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \sin 2y - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cos 2y.$$

The conjugate base \mathbb{F}_0^* has a column form $\mathbb{F}_0^* = \{f_1/2, f_2/2, f_3/2\}$. If $\mathbb{Q} = \{q_1, q_2, q_3\}$, then $\mathbb{F}_0^* \circ \mathbb{Q}$ denote a matrix with elements $\frac{1}{2\pi} \int_0^\pi f_j(y) q_k(y) dy$, $j, k \in \{1, 2, 3\}$. It is easy to check that $\mathbb{F}_0^* \circ \mathbb{F}_0 = I$. Now we shall use the algorithm from the Corollary 3.1 and make $\Lambda(\varepsilon)$ from the equation

$$(\mathcal{A}_0 + \varepsilon \mathcal{A}_1 + \varepsilon^2 \mathcal{A}_2)(\mathbb{F}_0 + \varepsilon \mathbb{F}_1 + \dots) = (\mathbb{F}_0 + \varepsilon \mathbb{F}_1 + \dots)(\Lambda_0 + \varepsilon \Lambda_1 + \dots)$$

From the first equation $\mathcal{A}_0 \mathbb{F}_0 = \mathbb{F}_0 \Lambda_0$ we find $\Lambda_0 = 0$. The second equation has the form

$$\mathcal{A}_0 \mathbb{F}_1 = \mathbb{F}_0 \Lambda_1 - \mathcal{A}_1 \mathbb{F}_0.$$

From the conditions for normal solvability we find

$$\Lambda_1 = \mathbb{F}_0^* \circ \mathcal{A}_1 \mathbb{F}_0 = \begin{bmatrix} 0 & 0 & h/2 \\ 0 & 0 & -\Delta \\ h/2 & \Delta & 0 \end{bmatrix}.$$

For $\Delta^2 < h^2/4$ we have $\sigma(\Lambda_1) = \{0, \pm \sqrt{h^2/4 - \Delta^2}\}$ i. e. the trivial solution (4.1) is not stable. This is a corollary of parametric resonance: with a small change of the frequency Δ and a small deviation (the coefficient for \dot{x} is proportional to ε^2), some of the solutions of (4.1) exponentially increase.

Now, let $\Delta = 0$ and $\{y(t), t \geq 0\}$ be a diffusion Markov process on $[0, 2\pi]$ with a C -infinitesimal operator

$$(\mathcal{L}v)(y) = 2 \frac{dv(y)}{dy} + \frac{\alpha^2}{2} \frac{d^2 v(y)}{dy^2}.$$

In this case, the number zero is a point of the specter of multiplicity one, and consequently, a base \mathbb{F}_0 consist of an unite matrix I . Let $\mathbb{F}(\varepsilon) = I + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$. Then $\Lambda_0 = 0$ and for finding f_1 and f_2 we have the equation

$$\mathcal{A}_0 f_1 = \Lambda_1 I - \mathcal{A}_1 I.$$

Using the condition of the orthogonality of the right-hand side to the kernel of the operator \mathcal{A}_0^* , in which there is only one element, we obtain $\Lambda_1 = 0$. Then from the equation

$$(\mathcal{A}_0 f_1)(y) = -(\mathcal{A}_1 f_0)(y). \quad (4.3)$$

we find $f_1(y)$. This solution has the form

$$f_1(y) = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \cos y + \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \sin y$$

and putting it in (4.3), it is possible to find all a_j and b_j . Further, for finding f_2 and Λ_2 we have the equation

$$\mathcal{A}_0 f_2 = \Lambda_2 I - \mathcal{A}_1 f_1 - \mathcal{A}_2 I.$$

Again, from the condition of normal solvability, we have

$$\Lambda_2 = -2\gamma + \frac{h}{2} a_2 = -2\gamma + h^2 \frac{\alpha^4 + 32}{\alpha^2(\alpha^4 + 64)}.$$

So, for $\gamma > \frac{h}{2} \frac{\alpha^4 + 32}{\alpha^2(\alpha^4 + 64)}$ a trivial solution is exponentially stable in the mean-square, disregarding the existence of a parametric resonance. It seems that for a stability, the existence of a "diffusion rushing" of a phase is useful - it allows to escape the resonance.

We shall give one of the possible method for the analyse of exponential p -stability of a stochastic oscillator, described by an equation

$$\ddot{x} + 2[\varepsilon \delta_1(y(t)) + \varepsilon^2 \delta_2(y(t))]\dot{x} + [1 + \varepsilon g_1(y(t)) + \varepsilon^2 g_2(y(t))]x = 0, \quad (4.4)$$

where $\{y(t), t \geq 0\}$ is a Markov process on a compact \mathbb{Y} with C -infinitesimal operator \mathcal{L} . Let the functions $\delta_1(y)$, $\delta_2(y)$, $g_1(y)$, $g_2(y)$ be continuous on \mathbb{Y} , $\varepsilon \in [0, \varepsilon_0]$. Let us make a substitution $x = r \cos \varphi$, $\dot{x} = -r \sin \varphi$ and use polar coordinate r and φ :

$$\dot{r} = r [\varepsilon a_1(\varphi, y(t)) + \varepsilon^2 a_2(\varphi, y(t))], \quad (4.5)$$

$$\dot{\varphi} = 1 + \varepsilon b_1(\varphi, y(t)) + \varepsilon^2 b_2(\varphi, y(t)), \quad (4.6)$$

where

$$a_j(\varphi, y) = -\delta_j(y)(1 - \cos 2\varphi) + \frac{1}{2} g_j(y) \sin 2\varphi,$$

$$b_j(\varphi, y) = -\delta_j(y) \sin 2\varphi + \frac{1}{2} g_j(y)(1 + \cos 2\varphi), \quad j=1,2.$$

Because of the periodicity of the right-hand side of (4.6) in φ with the period π , it is possible to consider a pair $\{\varphi(t), y(t), t \geq 0\}$ as a Markov process on the compact $\mathbb{Y} \times [0, \pi]$ with C -infinitesimal operator Γ , defined on $\mathcal{D}(\Gamma) \subset C(\mathbb{Y} \times [0, \pi])$ with

$$(\Gamma w)(\varphi, y) = (\Gamma_0 w)(\varphi, y) + \varepsilon(\Gamma_1 w)(\varphi, y) + \varepsilon^2(\Gamma_2 w)(\varphi, y), \quad (4.7)$$

where

$$(\Gamma_0 w)(\varphi, y) = \frac{\partial w(\varphi, y)}{\partial \varphi} + (\mathcal{L}_y w)(\varphi, y),$$

$$(\Gamma_j w)(\varphi, y) = b_j(\varphi, y) \frac{\partial w(\varphi, y)}{\partial \varphi}, \quad j=1,2,$$

and the operator \mathcal{L}_y is a C -infinitesimal operator of Markov process $\{y(t), t \geq 0\}$, taking effect on the elements $w(\varphi, y)$ of the space $C(\mathbb{Y} \times [0, \pi])$ on the second argument. We make a substitution $u = r^p$, for $p > 0$, in (4.5):

$$\dot{u} = u [\varepsilon p(a_1(\varphi(t), y(t)) + \varepsilon(a_2(\varphi(t), y(t))))]$$

and investigate the mean-square stability of the equation (4.8), using the result of preceding section for $n=1$. If we rewrite the operator \mathcal{A} of (2.4) for the equation (4.8), and use (4.7), then we get

$$(\mathcal{A}v)(\varphi, y) = (\mathcal{A}_0 v)(\varphi, y) + \varepsilon(\mathcal{A}_1 v)(\varphi, y) + \varepsilon^2(\mathcal{A}_2 v)(\varphi, y),$$

where $\mathcal{A}_0 = \Gamma_0$,

$$(\mathcal{A}_j v)(\varphi, y) = (\Gamma_j v)(\varphi, y) + 2pa_j(\varphi, y)v(\varphi, y), \quad j=1,2.$$

Since $\mathcal{D}(\Gamma_1) = \mathcal{D}(\Gamma_2) \subset \mathcal{D}(\Gamma_0)$, and the operators Γ_1 and Γ_2 are Γ_0 -limited on the graph of the operator Γ_0 [5], it follows that the family of operators $\{\mathcal{A}_0 + \varepsilon \mathcal{A}_1 + \varepsilon^2 \mathcal{A}_2\}$ is homomorphy at the point $\varepsilon=0$ [5]. Now it is possible to use the results of [5] for an analyse of Lyapunov equation (3.8) for $n=1$

$$((\mathcal{A}_0 + \varepsilon \mathcal{A}_1 + \varepsilon^2 \mathcal{A}_2)q)(\varphi, y) = -1 \quad (4.9)$$

If we exchange a little the proof of the Corollary 3.2, we shall give necessary and sufficient condition of the exponential stability in the mean-square of the trivial solution of the equation (4.8) in the form

$$\hat{q}^{\varepsilon}(\varphi, y) \stackrel{\text{def}}{=} \sum_{k=-d}^0 \varepsilon^k q_k(\varphi, y) > 0, \quad \forall \varphi \in [0, \pi], \quad \forall y \in \mathbb{V},$$

where

\hat{q}^{ε} is a main part of Maclaurin series for the solution (4.9). Now, we shall describe an algorithm for an analyse of the stability in the case $d < 2$. If we substitute (4.10) in (4.9) for $d=1$, then

$$\frac{\partial q_{-1}(\varphi, y)}{\partial \varphi} + (\mathcal{L}_y q_{-1})(\varphi, y) = 0, \quad (4.11)$$

$$\frac{\partial q_0(\varphi, y)}{\partial \varphi} + (\mathcal{L}_y q_0)(\varphi, y) = -(\mathcal{A}_1 q_{-1})(\varphi, y) - 1. \quad (4.12)$$

Since we want to confirm the condition for the normal solvability, we need a kernel of conjugate operator

$$-\frac{\partial g(\varphi, y)}{\partial \varphi} + (\mathcal{L}_y^* g)(\varphi, C) = 0, \quad (4.13)$$

where $g(\varphi, C)$ is a function of two arguments: $\varphi \in [0, \pi]$ and a set C from the σ -algebra $\Sigma_{\mathbb{V}}$ of Borel subsets on the compact \mathbb{V} . If the Markov process $\{y(t), t \geq 0\}$ is uniformly ergodic, a solution (4.13) must be an invariant measure $\mu(C)$ of this process, and a solution of the equation (4.11) can be expressed by method of separation of variables, using tensor product $C([0, \pi]) \otimes C(\wedge)$. If $\{y(t), t \geq 0\}$ is uniformly ergodic it follows $q_{-1}(\varphi, y) \equiv \bar{q}$ and the condition for the solvability of the second equation has the form

$$2p\bar{q} \frac{1}{\pi} \int_0^{\pi} \delta_1(y) \mu(dy) = -1.$$

From that, it is easy to obtain the condition of stability: $\bar{q} > 0$ or

$$\int_{\mathbb{V}} \delta_1(y) \mu(dy) > 0.$$

We got the result which was described in Corollary 3.1 in somewhat different situation: If a mean value (for stationary measure of the process $\{y(t)\}$) of diffusion coefficient $\delta_1(y)$ is positive, then the

trivial solution of (4.5) is exponentially stable in the mean-square, for each $p > 0$. Note that

$$\mathbb{E}\{|u(t)|^2\} = \mathbb{E}\{(|x(t)|^2 + |\dot{x}(t)|^2)^p\},$$

i. e. the exponential stability in mean-square of the trivial solution of (4.8) is equivalent to the exponentially $2p$ -stability of the trivial solution of (4.4). Now, let

$$\int_Y \delta_1(y) \mu(dy) = 0.$$

Then, instead of (4.11)-(4.12) we should use a system of equation

$$\frac{\partial q_{-2}(\varphi, y)}{\partial \varphi} + (\mathcal{L}_y q_{-2})(\varphi, y) = 0, \quad (4.14)$$

$$\frac{\partial q_{-1}(\varphi, y)}{\partial \varphi} + (\mathcal{L}_y q_{-1})(\varphi, y) = -(\mathcal{A}_1 q_{-2})(\varphi, y), \quad (4.15)$$

$$\frac{\partial q_{-1}(\varphi, y)}{\partial \varphi} + (\mathcal{L}_y q_0)(\varphi, y) = -(\mathcal{A}_1 q_{-1})(\varphi, y) - (\mathcal{A}_2 q_{-2})(\varphi, y) - 1. \quad (4.16)$$

In our case $q_{-2}(\varphi, y) \equiv \bar{q}$ and

$$(\mathcal{A}_1 \bar{q})(\varphi, y) = 2p\bar{q}\{-\delta_1(y)(1-\cos 2\varphi) + \frac{1}{2}g_1(y)\sin 2\varphi\}. \quad (4.17)$$

Let us find $q_{-1}(\varphi, y)$ from (4.15) (for $q_{-2}(\varphi, y) \equiv \bar{q}$), put it in (4.16) and determine conditions of exponentially $2p$ -stability of the trivial solution of (4.4) if the form $\bar{q} > 0$, where \bar{q} can be find from an equation

$$\begin{aligned} \frac{1}{\pi} \int_0^\pi \int_Y \left\{ b_1(\varphi, y) \frac{\partial q_{-1}(\varphi, y)}{\partial \varphi} + 2pa_1(\varphi, y)q_{-1}(\varphi, y) \right\} \mu(dy) d\varphi \\ + 2p\bar{q} \int_Y [-\delta_2(y) + \frac{1}{2}g_2(y)] \mu(dy) = -1. \end{aligned}$$

In order to solve (4.15), it is convenient to present $q_{-1}(\varphi, y)$ as a Fourier series of variable φ on the segment $[0, \pi]$ and the form of right-side (i. e. (4.17)). Note that in the first integral in (4.18) we have use only the first addend in

$$g_{-1}(\varphi, y) = D_0(y) + D_1(y)\cos 2\varphi + B_1(y)\sin 2\varphi + \dots$$

i. e. $a_1(\varphi, y)$ and $b_1(\varphi, y)$ have only those addends. As a conclusion we shall give a note about asymptotic stability of trivial solution of (4.4) with probability one. Using martingale methods, analogously [9], it is possible to prove that the exponentially p -stability with $p > 0$ guarantee an asymptotic stability with probability one. Then, for small enough $p > 0$ in (4.18) only the addends which have p in the first degree, determine the sign of number \bar{q} , i. e. this formula, for small $p > 0$, can be considered in a simpler form

$$\begin{aligned} & \frac{1}{\pi} \int_0^\pi \int_Y (2\delta_1(y)\cos 2\varphi + g_1(y)\sin 2\varphi) q_{-1}(\varphi, y) \mu(dy) d\varphi \\ & + 2pq \int_Y [-\delta_2(y) + \frac{1}{2} g_2(y)] \mu(dy) + O(p^2) = -1. \end{aligned} \quad (4.19)$$

From (4.15), for $\frac{1}{\pi pq} \int_0^\pi \cos 2\varphi q_{-1}(\varphi, y) d\varphi = z_1(y)$ and $\frac{1}{\pi pq} \int_0^\pi \sin 2\varphi q_{-1}(\varphi, y) d\varphi = z_2(y)$, we have a system of equations

$$\begin{aligned} 2z_2(y) + (\mathcal{L}_y z_1)(y) &= -\delta_1(y), \\ -2z_1(y) + (\mathcal{L}_y z_2)(y) &= -\frac{1}{2} g_1(y), \end{aligned} \quad (4.20)$$

From that, the conditions of asymptotic stability with probability one can be set in the form

$$\int_0^\pi \{2z(y)\delta(y) + z(y)g(y) + g(y) - 2\delta(y)\} \mu(dy) < 0.$$

Note that the conditions of uniform ergodicity guarantee the existence and uniqueness of the solution of the system (4.20), i. e. the specter of the operator \mathcal{L}_y , beside zero, does not have other points on the imaginary axe.

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O PARAMETARSKOJ SLUČAJNOJ REZONANSI

Koristeći metod Ljapunova, ispitivana je stabilnost trivijalnog rešenja linearne slučajne diferencijalne jednačine. Rezultati su primenjeni na problem stabilnosti slučajnog oscilatora.

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