Zbornik radova Filozofskog fakulteta u Nisu
Serija Matematika 5(1991), 23-37

Slavik V. Jablan

COLOR-SYMMETRY GROUPS OF FRIEZES

(Received 10.10.1990)

Using color-symmetry characteristics of symmetry groups, according to "strong" and "middle" equality criterion, all color-symmetry groups of friezes $G_{21}{}^{P}$, are derived.

1. Introduction.

The concept of P-symmetry (permutation symmetry) introduced by A.M.Zamorzaev [1,2] is defined as follows. If P is a subgroup of the symmetric permutation group of p indices, and G is a discrete symmetry group, every transformation $C=cS=Sc$, $c\in P$, $S\in G$ is a P-symmetry transformation. Every group G^P derived from G by subtituting its symmetries by the corresponding P-symmetries is a P-symmetry group. If the substitutions included in G^P exhaust the group P, G^P is a complete P-symmetry group. Every complete P-symmetry group G^P can be derived from its generating group G by searching in G and P for normal subgroups H and Q for which the isomorphism $G/H\cong P/Q$ holds, by paired multiplication of the cosets corresponding in this isomorphism, and by the unification of the products obtained. The groups of complete P-symmetry fall, respectively, into the senior ($G=H$ and $G^P=G\times P$), middle ($Q=P$, $Q=I$ and $I\subset Q\subset P$) and junior groups ($G/H\cong P$ and $G^P\cong G$). In this paper only the junior P-symmetry groups are considered.

There are few different criterions for the equality of

AMS Subject Classification (1980): Primary 20H15 junior P-symmetry groups. The most rafined ("strong") criterion is the following: let a color-permutation group P be decomposed in the product of different (irreducible) groups $P=P_1^{a_1}\ldots P_i^{a_i}$, where $H_1,\ldots,H_a$ $(a=a_1+\ldots+a_i)$ are the subgroups of G such that $G/H_1\cong P_1$, $G/H_2\cong P_1,\ldots,G/H_{a_1}\cong P_1,\ldots,$ $G/H_a\cong P_n$, and H is their section ($G/H\cong P$). In this case every P-symmetry group can be uniquely defined as $G/(H_1,\ldots,H_a)/H$ [2]. If subgroups which result in isomorphic quotient groups are taken as equivalent, or if only reduced symbols G/H are considered, two (sub)criterions (the "middle" and "weak"), are obtained. Knowing P-symmetry groups classified according to the "middle" criterion, they can be simply classified according to the "strong" criterion by permuting the equivalent subgroups.

If H is a normal subgroup of G, and P is the regular permutation group of the order N ($G/H\cong P$), then $[G:H]=N$. In the case of an irregular permutation group P, instead of a group/subgroup symbol G/H [1,2,3,4], a symbol $G/H_1/H$ will be used, where H_1 is the subgroup retaining invariant one index, and H is the symmetry subgroup of G^P [5]. Then, H_1 is not a normal subgroup of G, $[G:H_1]=N_1$, $[H_1:H]=N_2$, and $N=N_1N_2$.

Bohm symbols [6] are used to denote the corresponding categories of isometric symmetry groups. In a symbol $G_n\ldots$ the first subscript n is the maximal dimension of space in which the transformations of the symmetry group act, while the following subscripts are the maximal dimensions of subspaces invariant with respect to the action of the symmetry group and properly included in each other. The corresponding categories of P-symmetry groups are denoted by additional P-superscripts.

The indices ascribed to the points of a figure with a P-

symmetry group have an extrageometric sense with respect to the space in which the figure is considered. In additional dimensions such index permutations can be geometrically interpreted, making possible an investigation of multi-dimensional symmetry groups by means of P-symmetry groups [7]. A simple example illustrating this is the derivation of the symmetry groups of bands G_{321} from the symmetry groups of friezes G_{21}. The 31 symmetry group of bands G_{321} can be derived from the symmetry groups of friezes G_{21} by using antisymmetry (31 $G_{321} \simeq$ 7 generating G + 7 senior ($G\times\underline{1}$) + 17 junior G^1, $G \in G_{21}$). The same method can be used in order to derive the P-symmetry groups of bands from the P-symmetry groups of friezes [1,2,8].

If $P = C_2{}^l$, according to the "strong" criterion we have the simple ($l=1$) and multiple ($l \geq 2$) antisymmetry groups, and according to the "middle" criterion, so-called Mackay groups (or compound groups) [9,10]. In the first case, all quotient groups C_2 are treated as non-equivalent, and in the second case as the equivalent ones. Hence, in the first case besides of their choice, there are considered permutations of the corresponding subgroups of the index 2.

Let a symmetry group G be given by the presentation:

$\{S_1, S_2, \ldots, S_r\}$ $\qquad g_k(S_1, S_2, \ldots, S_r) = I,$ $\qquad k = 1, 2, \ldots, s,$

and let $e_1, e_2, \ldots, e_l$ be the antiidentity transformations of the first, second, ..., lth kind satisfying the relations:

$$e_i{}^2 = I, \qquad e_i e_j = e_j e_i, \qquad e_i S_q = S_q e_i,$$

$$i, j \in \{1, 2, \ldots, l\}, \quad q \in \{1, 2, \ldots, r\}.$$

Every transformation $S' = e'S$, $S \in G$, where e' is an antiidentity transformation or their product is called a (multiple) antisymmetry transformation. Every group G' derived from G, which contains at least one (multiple) antisymmetry transformation is called a (multiple) antisymmetry group, and the group G is called its generating group. A junior multiple antisymmetry group is called the M^m-type group if its corresponding antiidentity transformations are independent, this means, if they are not representable by the others. In the theory of simple and multiple antisymmetry [9] only the derivation of junior simple and multiple antisymmetry groups of the M^m-type is non-trivial. They can be derived very efficiently by using the antisymmetric characteristic method [11,12].

In particular, for $l=1$ we have the (simple) antisymmetry. Among (simple) antisymmetry groups we can distinguish the senior antisymmetry groups of the form $G\times\{e_1\}$, with the structure $G\times C_2$, where $\{e_1\}$ denotes the group generated by e_1, with the structure C_2, and the junior antisymmetry groups isomorphic to their generating group G. Since the antiidentity transformation e_1 can be identified with the (hyper)reflection in the invariant $(n+1)$-plane, to every antisymmetry group of the category $G_n\ldots{}^1$ corresponds the symmetry group of the category $G_{(n+1)n\ldots}$. Namely, to every senior antisymmetry group $G\times\{e_1\}$ corresponds the symmetry group $G\times\{T_1\}$, where $\{T_1\}$ denotes the symmetry group with the structure D_1 ($D_1\cong C_2$), generated by a plane reflection T_1, and to every junior antisymmetry group G^1 given by the presentation:

$\{S_1{}^1, S_2{}^1, \ldots, S_r{}^1\} \quad g_k(S_1{}^1, S_2{}^1, \ldots, S_r{}^1)=I, \quad k=1,2,\ldots,s,$

where the set $\{S_1{}^1, S_2{}^1, \ldots, S_r{}^1\}$ consists of the (anti)generators $S_q{}^1$ ($S_q{}^1=S_q$ or $S_q{}^1=e_1S_q$, $q=1,2,\ldots,r$), corresponds the $(n+1)$-dimensional symmetry group generated by the symmetries $S_q{}^1=S_q$ or $S_q{}^1=T_1S_q$.

In the case of Belov (p)-symmetry [2] (or C_p-symmetry [4]), the group $P\cong C_p$ is generated by the permutation $c_1=(12...p)$ satisfying the relations:

$c_1{}^p=I \quad c_1 S=Sc_1, \quad S\in G.$

In the case of Pawley (p')-symmetry [2] (or $D_{p(2p)}$-symmetry [4]), the group $P\cong D_{p(2p)}$ is the regular dihedral permutation group generated by the permutations c_1 and $e_1=(11')$ satisfying the relations:

$c_1{}^p=e_1{}^2=(c_1 e_1)^2=I \quad c_1 S=Sc_1 \quad e_1 S=Se_1, \quad S\in G.$

In the case of ($p2$)-symmetry [2] (or D_p'-symmetry [4]), the group $P\cong D_p$ is the irregular dihedral permutation group generated by the permutations c_1 and $e_1=(12)$ satisfying the same relations.

All remaining P-symmetries which the symmetry groups of friezes admit, are the combinations of C_p-, $D_{p(2p)}$- and D_p-symmetry with the simple and multiple antisymmetry.

2. Color-symmetry groups of friezes.

The seven symmetry groups of friezes, denoted by reduced symbols [13], are given by their presentations and structure [11,14]:

11	$\{X\}$			C_∞
1g	$\{P\}$			C_∞
12	$\{X,T\}$	$T^2=(TX)^2=E$		D_∞
	$\{T,T_1\}$	$T^2=T_1{}^2=E$	$(T_1=TX)$	
m1	$\{X,R_1\}$	$R_1{}^2=(R_1 X)^2=E$		D_∞
	$\{R_1,R_2\}$	$R_1{}^2=R_2{}^2=E$	$(R_2=R_1 X)$	
mg	$\{P,R_1\}$	$R_1{}^2=(R_1 P)^2=E$		D_∞
	$\{R_1,T\}$	$R_1{}^2=T^2=E$	$(T=R_1 P)$	

1m	$\{X, R\}$	$R^2=E$	$RX=XR$			$C_\infty \times C_2$
mm	$\{X, R, R_1\}$	$R^2=R_1{}^2=(R_1X)^2=E$	$XR=RX$	$RR_1=R_1R$		$D_\infty \times C_2$
	$\{R, R_1, R_2\}$	$R^2=R_1{}^2=R_2{}^2=E$	$RR_1=R_1R$	$RR_2=R_2R$	$(R_2=R_1X)$.	

With regard to their structure, 7 symmetry groups of friezes can be divided in 4 classes:

1) 11, 1g of the structure C_∞;

2) 12, m1, mg of the structure D_∞;

3) 1m of the structure $C_\infty \times C_2$; and

4) mm of the structure $D_\infty \times C_2$.

Having in mind the reducibility of the groups C_{4k-2} ($C_{4k-2}=C_{2k-1}\times C_2$) and D_{4k-2} ($D_{4k-2}=D_{2k-1}\times C_2$), from this we can conclude that P can be, respectively:

1) C_k;

2) D_k, $D_{k(2k)}$;

3) C_k, $C_k \times C_2$;

4) D_k, $D_{k(2k)}$, $D_k \times C_2$, $D_{k(2k)} \times C_2$, $D_{2k-1(4k-2)} \times C_2{}^2$.

<u>*Definition 1*</u> Let the set of elements of a symmetry group G be divided in equivalency classes consisting of elements equivalent with regard to symmetry. The resulting system is called the color-symmetry characteristic $CC(G)$ of the group G.

<u>*Theorem 1*</u> Two P-symmetry groups derived from the same generating symmetry group G are equal *iff* they possess equal color-symmetry characteristics.

<u>*Theorem 2*</u> From two symmetry groups G and G_1 which possess isomorphic color-symmetry characteristics $CC(G)\cong CC(G_1)$ the same number of color-symmetry (P-symmetry) groups can be derived. All

P-symmetry groups derived from G and G_1 are corresponding in this CC-isomorphism.

According to Definition 1, the (reduced) color-symmetry characteristics of the symmetry groups of friezes are the following:

$i)$ 11 $\{X\}$ $CC:\{\ldots X^{-3}, X^{-2}, X^{-1}, E, X, X^2, X^3 \ldots\}$

$ii)$ 1g $\{P\}$ $CC:\{\ldots P^{-3}, P^{-1}, P, P^3 \ldots\}$

$iii)$ 12 $\{X, T\}$ $CC:\{\ldots TX^{-2}, TX^{-1}, T, TX, TX^2 \ldots\}$

$\{T, T_1\}$ $CC:\{\ldots T(TT_1)^{-2}, T(TT_1)^{-1}, T_1, T_1, T(TT_1)^2 \ldots\}$

m1 $\{X, R_1\}$ $CC:\{\ldots R_1X^{-2}, R_1X^{-1}, R_1, R_1X, R_1X, R_1X^2 \ldots\}$

$\{R_1, R_2\}$

$CC:\{\ldots R_1(R_1R_2)^{-2}, R_1(R_1R_2)^{-1}, R_1, R_2, R_1(R_1R_2)^2 \ldots\}$

$iv)$ mg $\{P, R_1\}$

$CC:\{\ldots R_1P^{-4}, R_1P^{-2}, R_1, R_1P^2, R_1P^4 \ldots\}\{\ldots R_1P^{-3}, R_1P^{-1}, R_1P, R_1P^3 \ldots\}$

$\{R_1, T\}$ $CC:\{\ldots R_1(R_1T)^{-4}, R_1(R_1T)^{-2}, R_1, R_1(R_1T)^2, R_1(R_1T)^4 \ldots\}$
$\{\ldots R_1(R_1T)^{-3}, R_1(R_1T)^{-1}, T, R_1(R_1T)^3 \ldots\}$

$v)$ 1m $\{X, R\}$ $CC:\{R\}\{\ldots X^{-3}, X^{-2}, X^{-1}, E, X, X^2, X^3 \ldots\}$

$vi)$ mm $\{X, R, R_1\}$ $CC:\{R\}\{\ldots R_1X^{-2}, R_1X^{-1}, R_1, R_1X, R_1X, R_1X^2 \ldots\}$

$\{R, R_1, R_2\}$

$CC:\{R\}\{\ldots R_1(R_1R_2)^{-2}, R_1(R_1R_2)^{-1}, R_1, R_2, R_1(R_1R_2)^2 \ldots\}$

With regard to the CC-isomorphism, the symmetry groups of friezes belonging to the classes 1-4 are distributed in 6 subclasses, denoted by i-vi. Consequently, the derivation of all P-symmetry groups of friezes is reduced to the derivation of P-symmetry groups from 6 symmetry groups of friezes, i.e. from the representatives of the subclasses mentioned: 11, 1g, 12, mg, 1m and mm. From them, all P-symmetry groups of friezes classified according to the "middle" equality criterion (and resulting "strong" and "weak" criterion), are derived.

3. Catalogue of color-symmetry groups of friezes

In Table 1.1- 1.3 Mackay groups (or compound groups [10]) of friezes with $P \cong C_2^{\,l}$ ($1 \le l \le 3$), derived from the groups-representatives (11,1g,12,mg,1m,mm), are given. The multiple antisymmetry groups of friezes [9] can be simply derived from them by permuting the equivalent subgroups of the index 2. The results according to the "strong" and "weak" criterion are given at the end of each table.

Table 1.1.

C_2				
11/11	12/12	mg/m1	1m/1m	mm/mm
	12/11	mg/12	1m/1g	mm/1m
1g/11		mg/1g	1m/11	mm/mg
				mm/m1
				mm/12

There are 17 $G_{21}^{\,1}$.

Table 1.2

$C_2^{\,2}$		
12/(12,12)/11	1m/(1m,1g)/11	mm/(mm,mg)/12
12/(12,11)/11	1m/(1m,11)/11	mm/(mm,12)/12
	1m/(1g,11)/11	mm/(mg,12)/12
		mm/(1m,mg)/1g
mg/(m1,12)/11	mm/(mm,mm)/1m	mm/(mg,mg)/1g
mg/(m1,1g)/11	mm/(mm,1m)/1m	mm/(1m,m1)/11
mg/(12,1g)/11	mm/(mm,mg)/m1	mm/(1m,12)/11
	mm/(mm,m1)/m1	mm/(m1,12)/11
	mm/(mg,m1)/m1	

From 23 compound groups of the M^2-type (or $2\underline{2}$-symmetry groups [2]) can be derived 2x19+1x4=42 multiple antisymmetry groups of the M^2-type, and 9 $D_{2(4)}$-symmetry (or (2')-symmetry [2]) groups, according to the "weak" equality criterion (G/H).

For l=2,3 the multiple antisymmetry groups can be obtained from Mackay groups given in Table 1.2- 1.3 by permuting the corresponding equivalent subgroups of the index 2. In non-exceptional cases, the number of multiple antisymmetry groups derived from a Mackay group will be *7!* in the case that all subgroups mentioned are different, or *7!/2* if two of them are identical. In exceptional cases (denoted by +) the section of two identical subgroups with a remaining subgroup results in distinct subgroups of the index 4, so these identical subgroups are not equivalent, and the number of multiple antisymmetry groups derived will be *7!*. If a Mackay group (Table 1.3) is not uniquely defined by its extended group/subgroup symbol, an additional symbol (given in parentheses (), indicating to Table 1.2), which points to the section of subgroups of the index 2, is given.

Table 1.3

C_2^3

mm/(mm,mm,mg)/11+	mm/(mm,1m,12)/11	mm/(1m,mg,m1)/11
mm/(mm,mm,m1)/11	mm/(mm,mg,mg)/11+	mm/(1m,mg,12)/11
mm/(mm,mm,12)/11	mm/(mm,mg,m1)/11	mm/(mg,mg,m1)/11
mm/(mm,1m,mg)(m1)/11	mm/(mm,mg,12)/11	mm/(mg,mg,12)/11
mm/(mm,1m,mg)(12)/11	mm/(mm,m1,12)/11	mm/(mg,m1,12)/11
mm/(mm,1m,m1)/11		

From 16 compound groups of the M^3-type (or $22\underline{1}$-symmetry groups [2]) can be derived 6x12+3x4=84 multiple antisymmetry groups of the M^3-type, and the only one group G/H.

The remaining groups G_{21}^{P} are classified in the same way, according to P-symmetries ($G/H \cong P$), using the "middle" equality criterion. In Table 2.1 – 2.7 every P-symmetry group is given by its (extended) group/subgroup symbol and by the number n. Each table is followed by data about the possible reducibility of the group P, the number of the P-symmetry groups for the given n, the number of crystallographic P-symmetry groups of friezes [1,2], by results corresponding to the "strong" equality criterion (which can be simply derived by permuting the equivalent subgroups of the index 2), and to the "weak" equality criterion (G/H). Owing to the reducibility of the groups C_{4k-2} and D_{4k-2}, $P \cong C_{4k-2} \times C_2^{l-1} = C_{2k-1} \times C_2^{l}$ and $P \cong D_{4k-2} \times C_2^{l-1} = D_{2k-1} \times C_2^{l}$ ($l \geq 1$). Therefore, the cases $P \cong C_{2k-1} \times C_2^{l}$, $D_{2k-1} \times C_2^{l}$ are not considered.

Table 2.1

$\underline{C_n}$

11/11	k	1m/1m	k
		1m/1g	$2k$
1g/1g	$2k-1$	1m/11	$4k-2$
1g/11	$2k$		

$C_{4k-2} = C_{2k-1} \times C_2$

k	2	2	5
$2k-1$	1	3	3
$2k$	2	4	4
$4k-2$	1	6	5

Table 2.2

$\underline{D_{n(2n)}}$

12/11	k	mm/1m	k
		mm/1g	$2k$
mg/1g	$2k-1$	mm/11	$4k-2$
mg/11	$2k$		

k	3	2	6
$2k-1$	1	3	4
$2k$	2	4	5
$4k-2$	1	6	6

Table 2.3

$\underline{D_n}$ ($n \geq 2$)

12/12/11	k ($k \geq 3$)	mm/mm/1m	k ($k \geq 3$)
		mm/mg/1g	$2k$
mg/mg/1g	$2k-1$	mm/m1/11	$4k-2$
mg/m1/11	$2k$	mm/12/11	$4k-2$
mg/12/11	$2k$		

$D_{4k-2} = D_{2k-1} \times C_2$

k	3	3	4
$2k-1$	1	4	6
$2k$	3	6	8
$4k-2$	2		

The transition from the "middle" (in this case coinciding to the "strong") equality criterion ($G/H_1/H$) to the "weak" equality criterion (G/H), results in the transition from D_n-symmetry (or $p2$-symmetry [2]) to $D_{n(2n)}$-symmetry (or p'-symmetry [2]).

Table 2.4

<u>$C_n \times C_2$</u>

1m/(1m,1g)/11	$2k$	1m/(11,1g)/11	$4k-2$
1m/(1m,11)/11	$2k$	1m/(11,1m)/11	$4k-2$
1m/(1g,11)/11	$2k$	1m/(1g,1m)/11	$4k-2$
1m/(1g,1g)/11	$4k$		

$C_{4k-2} = C_{2k-1} \times C_2$

$2k$	3	2	6
$4k$	1	4	4
$4k-2$	3	6	6

In transition from the "middle" (in this case coinciding to the "strong") equality criterion ($G/(H_1,H_2)/H$) to the "weak" equality criterion (G/H), we have the following results:

$2k$	1	2	1
		4	1
		6	1

Table 2.5

<u>$D_{n(2n)} \times C_2$</u>

12/(11,12)/11	$2k-1$	mm/(1m,12)/11	k
		mm/(1m,mg)/11	$2k$
mg/(1g,m1)/11	$2k-1$	mm/(1g,m1)/11	$2k$
mg/(1g,12)/11	$2k-1$	mm/(1g,12)/11	$2k$
		mm/(1g,mg)/11	$4k$
mm/(1m,mm)/1m	$2k-1$	mm/(1g,mm)/11	$4k-2$
mm/(1m,mg)/1g	$2k-1$	mm/(11,mm)/11	$4k-2$
mm/(1m,m1)/11	k	mm/(11,mg)/11	$4k-2$

$2k-1$	6	3	8
k	2	4	6
$2k$	3	6	8
$4k-2$	3		
$4k$	1		

In the transition from the "middle" ($G/(H_1,H_2)/H$) to the "weak" equality criterion (G/H), we have the following results:

$2k-1$	5	3	6
k	1	4	1
		6	1

Table 2.6

$D_n \times C_2$

mm/(mm/1m,mg)/m1/11	$2k$	mm/(mg/1g,mg)/12/11	$4k$
mm/(mm/1m,mg)/12/11	$2k$	mm/(mg/1g,mm)/m1/11	$4k-2$
mm/(mm/1m,m1)/m1/11	$2k$	mm/(mg/1g,mm)/12/11	$4k-2$
mm/(mm/1m,12)/12/11	$2k$	mm/(m1/11,mm)/m1/11	$4k-2$
mm/(mg/1g,m1)/m1/11	$2k$	mm/(m1/11,mg)/m1/11	$4k-2$
mm/(mg/1g,12)/12/11	$2k$	mm/(12/11,mm)/12/11	$4k-2$
mm/(mg/1g,mg)/m1/11	$4k$	mm/(12/11,mg)/12/11	$4k-2$

$D_{4k-2} \times C_2 = D_{2k-1} \times C_2{}^2$

$2k$	6	4	8
$4k$	2	6	12
$4k-2$	6		

In transition from the "middle" (in this case coinciding with the "strong") equality criterion ($G/(H_4/H_3, H_2)/H_1/H$) to the "weak" equality criterion ($G/H_1/H$), we have the following results:

$2k$	2	4	2
		6	2

If the symbols $G/H_1/H$ are reduced to G/H, then:

$2k$	1	4	1
		6	1

Table 2.7

Using some additional data pointing to the section of the corresponding subgroups, every extended group/subgroup symbol uniquely defines the colored symmetry group. By additional data given in parentheses () are denoted sections indicating to Table 1.2.

$D_{n(2n)} \times C_2{}^2$			
mm/(1m,mm,mg)(m1)/11	$2k-1$	mm/(1m,mm,12)/11	$2k-1$
mm/(1m,mm,mg)(12)/11	$2k-1$	mm/(1m,mg,m1)/11	$2k-1$
mm/(1m,mm,m1)/11	$2k-1$	mm/(1m,mg,12)/11	$2k-1$

$2k-1$	6	3	6

In transition from the "middle" to the "strong" equality criterion, by permuting the corresponding subgroups of the index 2, from every group two groups can be derived.

According to the "weak" equality criterion (G/H), we have the following results:

$2k-1$	1	3	1

The results obtained imply the complete survey of the group/subgroup relations of the symmetry groups of friezes. The Table 3.1 giving the index of a subgroup in the group coincides with Table I [4]. In Table 3.2, giving the quotient group $P \cong G/H$ some corrections of Coxeter results [4. Table II], are included.

In Table 3.2 the group C_2 ($C_2 \cong D_1$) can be replaced, respectively, by D_N, where $[G:H]=N$ (Table 2.3). The *italic* indexes in Table 3.2 indicate that H is not a normal subgroup of G.

Table 3.1

	11	1g	12	m1	mg	1m	mm
11	k						
1g	2k	2k−1					
12	2k		*k*				
m1	2k			*k*			
mg	4k	4k−2	*2k*	*2k*	*2k−1*		
1m	2k	2k				k	
mm	4k	4k	*2k*	*2k*	*2k*	2k	*k*

Table 3.2

	11	1g	12	m1	mg	1m	mm
11	C_k						
1g	C_{2k}	C_{2k-1}					
12	$D_{k(2k)}$		C_2				
m1	$D_{k(2k)}$			C_2			
mg	$D_{2k(4k)}$	$D_{2k-1(4k-2)}$	C_2	C_2	D_{2k-1}		
1m	$C_k \times C_2$	C_{2k}				C_k	
mm	$D_{k(2k)} \times C_2$	$D_{2k(4k)}$	C_2	C_2	C_2	$D_{k(2k)}$	C_2

References:

[1] ZAMORZAEV A.M., PALISTRANT A.F., GALYARSKI E.I.: Tsvetnaya simmetriya, eyo obobscheniya i prilozeniya, *Shtiintsa, Kishinev*, 1979.

[2] ZAMORZAEV A.M., KARPOVA Yu.S., LUNGU A.P., PALISTRANT A.F.: *P*-simmetriya i eyo dal'neishee razvitie, *Shtiintsa, Kishinev*, 1986.

[3] SHUBNIKOV A.V., KOPTSIK V.A.: Symmetry in Science and Art, *Plenum Press, New York*, 1974.

[4] ČOXETER H.S.M.: *A Simple Introduction to Colored Symmetry*, I.J.Quantum Chemistry XXXI (1987), 455-461.

[5] KOPTSIK V.A., KOTZEV J.N.: *K teorii i klassifikatsii grupp tsvetnoi simmetrii. I. P-simmetriya*, Soobscheniya OIYaI, P4-8067, *Dubna*, 1974.

[6] BOHM J., DORNBERGER-SCHIFF K.: *The Nomenclature of Crystallographic Symmetry Groups*, Acta Cryst. A21 (1966),1004-1007.

[7] PALISTRANT A.F.: *Application of Two-Dimensional and Three-Dimensional Crystallographic P-symmetry Groups to the Study of Multi-Dimensional Symmetry Groups*, Colloquia Mathematica Societatis János Bolyai, *48*.Intuitive Geometry, Siófok, 1985, 465-479.

[8] JARRATT J.D., SCHWARZENBERGER R.L.E.: *Coloured Frieze Groups*, Utilitas Math. 19 (1981), 295-303.

[9] ZAMORZAEV A.M.: Teoriya prostoi i kratnoi antisimmetrii, *Shtiintsa, Kishinev*, 1976.

[10] MACKAY A.L.: *Extensions of Space Group Theory*, Acta Cryst. 10 (1957), 543-548.

[11] YABLAN S.V.: *Gruppy prostoi i kratnoi antisimmetrii bordyurov*, Publ. Inst. Math. (Belgrade) 36 (50) (1984), 35-44.

[12] JABLAN S.V.: *Algebra of Antisymmetric Characteristics*, Publ. Inst. Math. (Belgrade) 47 (61) (1990), 39-55.

[13] SENECHAL M.: *Point Groups and Color Symmetry*, Z. Kristall. 142 (1975), 1-23.

[14] JABLAN S.V.: Theory of Symmetry and Ornament, *The Mathematical Institute, Belgrade*, 1990 (to appear).

Slavik V. Jablan

GRUPE KOLORNE SIMETRIJE BORDURA

Primenom kolorno-simetrijskih karakteristika grupa simetrije, sve grupe kolorne simetrije bordura $G_{21}{}^{P}$ su izvedene u skladu sa "jakim" i "srednjim" kriterijumom jednakosti.

Slavik V. Jablan
Department of Mathematics
Phylosophical Faculty
18000 Niš
Ćirila i Metodija 2
Yugoslavia