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ENANTIOMORPHISM OF NON-CRYSTALLOGRAPHIC SIMPLE AND MULTIPLE ANTISYMMETRY GROUPS OF RODS

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Investigation of enantiomorphism of crystallographic l-multiple antisymmetry groups of rods [1] is extended on non-crystallographic groups of same category.

A natural extension of the (simple) antisymmetry ($l=1$) is the l-multiple antisymmetry ($l\geq 2$), introduced by A.M.Zamorzaev [2], where besides a generating symmetry group S we have the permutation group $P=C_2{}^l$ generated by l antiidentity transformations e_i ($i=1,2,\ldots,l$) satisfying the relations $e_i{}^2=E$, commuting between themselves and with all elements of S. All l-multiple antisymmetry groups fall into the three types: senior (S^k-), middle (S^kM^m-) and junior (M^m-type) multiple antisymmetry groups, where only the last ones, isomorphic to S, are non-trivial in the sense of derivation.

If a discrete symmetry group S is given by its presentation, for a derivation of the M^m-type groups the antisymmetric characteristic method (AC-method), can be used [1,3,4,5].

Definition 1. Let all products of generators of a symmetry group S, within which every generator participates once at the

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most, be given, and then subsets of transformations equivalent with regard to symmetry, be separated. The resulting system is called the antisymmetric characteristic of S ($AC(S)$).

Theorem 1. Two groups S' and S'' of the M^m -type derived from the same symmetry group S are equal *iff* they posses equal antisymmetric characteristics.

Theorem 2. Two symmetry groups S and S_1 with isomorphic antisymmetric characteristics generate the same number of the M^m -type groups for every fixed m ($1 \leq m \leq l$), wich correspond to each other with regard to structure.

According to Theorem 2, a complete information about all antisymmetric characteristics considered in this paper, can be obtained from the catalogue of non-isomorphic antisymmetric characteristics ($l \leq 4$) [3].

For denoting caregories of symmetry groups, Bohm symbols $G_r\ldots$ are used. Every category of symmetry groups of the space E^r is defined by the sequence $r\ldots$ of maximal invariant (sub)spaces inserted into one another in succession. The $G_r\ldots^l$ denotes the category of l-multiple antisymmetry groups derived from the category $G_r\ldots$.

Two l-multiple antisymmetry groups S' and S'' are equal if $S''=aS'a^{-1}$ for some orientation preserving affine transformation a. Two l-multiple antisymmetry groups S' and S'' are enantiomorphic if $S''=aS'a^{-1}$ holds only for some orientation reversing affine transformation(s) a.

All symmetry groups of rods G_{31} (without crystallographic restriction) can be distributed in 15 infinite classes [4,5]. From the six of them: 1) $p(2k+1)_j$ and $p(2k+1)_{2k+1-j}$ ($k \in N$, $1 \leq j \leq k$), 2) $p(2k)_{2j-1}$ and $p(2k)_{2k-2j+1}$ ($2 \leq k$, $1 \leq j \leq [k/2]$), 3) $p(2k+1)_j2$ and

$p(2k+1)_{2k+1-j}2$ ($k\in N$, $1\le j\le k$), 4) $p(2k)_{2j-1}22$ and $p(2k)_{2k-2j+1}22$ ($2\le k$, $1\le j\le [k/2]$), 5) $p(2k)_{2j}$ and $p(2k)_{2k-2j}$ ($2\le k$, $1\le j\le [k/2]$), 6) $p(2k)_{2j}22$ and $p(2k)_{2k-2j}22$ ($2\le k$, $1\le j< k/2$), for every fixed k and j, results an enantiomorphic group pair. By the results [1,2,6,7], it is understood and proved as well, that from every symmetry group belonging to an enantiomorphic group pair, the same number of simple and multiple antisymmetry groups is derived, and to every group of one family corresponds the enantiomorphic group of the other family. In works of Kishinev school the registration of enantiomorphic pairs among the M^m-type groups generated from them is realised by comparing International symbols of the same group obtained by Shubnikov-Zamorzaev and Belov method. This problem can be very efficiently solved using the AC-method [1].

The mentioned 6 infinite classes of enantiomorphic groups G_{31} are given in the International notation and notation proposed by A.M.Zamorzaev [2], each followed by the corresponding antisymmetric characteristic $AC(S)$, existential condition for the M^m-type groups [1], number of AC-isomorphism class and numbers $N_m(S)$ [3]:

1) $p(2k+1)_j$, $\{c\}(jc/2k+1(2k+1))$, AC: $\{jc/2k+1(2k+1)\}$,
   $c=jc/2k+1(2k+1)$;
   $p(2k+1)_{2k+1-j}$, $\{c\}(-jc/2k+1(2k+1))$, AC: $\{-jc/2k+1(2k+1)\}$,
   $c=-jc/2k+1(2k+1)$; 1.1, $N_1=1$;
2) $p(2k)_{2j-1}$, $\{c\}((2j-1)c/2k(2k))$, AC: $\{(2j-1)c/2k(2k)\}$,
   $c\neq c'$;
   $p(2k)_{2k-2j+1}$, $\{c\}(-(2j-1)c/2k(2k))$, AC: $\{-(2j-1)c/2k(2k)\}$,
   $c\neq c'$; 1.1, $N_1=1$;

3) $p(2k+1)_j2$, $\{c\}(jc/2k+1(2k+1):2)$, $AC:\{2,jc/2k+1(2k+1)2\}$,

$c=jc/2k+1(2k+1)$;

$p(2k+1)_{2k+1-j}2$, $\{c\}(-jc/2k+1(2k+1):2)$, $AC:\{2,-jc/2k+1(2k+1)2\}$,

$c=-jc/2k+1(2k+1)$; 2.2, $N_1=2$, $N_2=3$;

4) $p(2k)_{2j-1}22$, $\{c\}((2j-1)c/2k(2k):2)$, $AC:\{2,(2j-1)c/2k(2k)2\}$,

$c\neq c'$;

$p(2k)_{2k-2j+1}22$, $\{c\}(-(2j-1)c/2k(2k):2)$, $AC:\{2,-(2j-1)c/2k(2k)2\}$,

$c\neq c'$; 2.2, $N_1=2$, $N_2=3$;

5) $p(2k)_{2j}$, $\{c\}(jc/k(2k))$, $AC:\{c\}\{jc/k(2k)\}$;

$p(2k)_{2k-2j}$, $\{c\}(-jc/k(2k))$, $AC:\{c\}\{-jc/k(2k)\}$;

2.1, $N_1=3$, $N_2=6$;

6) $p(2k)_{2j}22$, $\{c\}(jc/k(2k):2)$,

$AC:\{jc/k(2k)\}\{\{2,2c\},\{jc/k(2k)2,jc/k(2k)2c\}\}$;

$p(2k)_{2k-2j}22$, $\{c\}(-jc/k(2k):2)$,

$AC:\{jc/k(2k)\}\{\{2,2c\},\{-jc/k(2k)2,-jc/k(2k)2c\}\}$,

3.4, $N_1=4$, $N_2=15$, $N_3=42$.

In every enumerated enantiomorphic group pair 1-6, its belonging groups differ only by the orientation of screws with the corresponding screw axes. The product of a screw motion corresponding to the first group with the translation $\pm c$ is the screw motion corresponding to the other group belonging to the enantiomorphic group pair. In this way, the multiplication of any screw motion from one of the 6 enumerated pairs with the translation $\pm c$ results in the transition from the discussed group to its enantiomorph.

This is illustrated by example of the M^1 -type groups of families with the generating groups enumerated by 5 ($p(2k)_{2j}$ and $p(2k)_{2k-2j}$). In accordance to the proposed method, every M^1-type group is followed by its antisymmetric characteristic AC and transformed antisymmetric characteristic AC^*, i.e. by the corresponding antisymmetric characteristic of the second family. Since we are dealing only with the antiidentities or their products, the signs ± are omitted.

$p(2k)_{2j}$, $\{c\}(jc/k(2k))$,

$AC:\{c\}\{jc/k(2k)\}$, $AC^*:\{c\}\{jc/k(2k)c\}$

$m=1$

1)	$\{c'\}(jc/k(2k))$,	$\{e_1\}\{E\}$,	$\{e_1\}\{e_1\}$,	1')
2)	$\{c\}(jc/k(2k)')$,	$\{E\}\{e_1\}$,	$\{E\}\{e_1\}$,	2')
3)	$\{c'\}(jc/k(2k)')$,	$\{e_1\}\{e_1\}$,	$\{e_1\}\{E\}$,	3')

$p(2k)_{2k-2j}$, $\{c\}(-jc/k(2k))$,

$AC:\{c\}\{-jc/k(2k)\}$,

1')	$\{c'\}(-jc/k(2k)')$,	$\{e_1\}\{e_1\}$;
2')	$\{c\}(-jc/k(2k)')$,	$\{E\}\{e_1\}$;
3')	$\{c'\}(-jc/k(2k))$,	$\{e_1\}\{E\}$.

The same procedure is used for finding the enantiomorph of the given M^m-type group ($m\geq 2$). This is realized by finding the group with the antisymmetric characteristic identical to in such a manner transformed antisymmetric characteristic of the initial group. We find such a group among the M^m-type groups derived from the second group of the discussed pair. These two groups make the M^m-type enantiomorphic group pair derived from the two enantiomorphic groups. In the discussed families to every of 6 M^2-type groups from the first, corresponds the enantiomorphic group from the second family.

Classifying in that manner all M^m-type groups derived from the 6 enantiomorphic group pairs, it is possible to acnowledge that all M^m-type groups of the families derived from enantiomorphic generating groups are mutually enantiomorphic. Therefore, if we add the numbers $N_i(S)$ $(1 \leq i \leq m)$, previously quoted for every enantiomorphic group pair 1-6, we can conclude that among M^m-type groups derived from them there are 13 M^1-type, 27 M^2-type and 42 M^3-type enantiomorphic group pairs for every fixed k and j.

The total number of all different enantiomorphic pairs among all simple and multiple antisymmetry groups of rods is not exhausted by the given numbers. Except from the 6 infinite classes aforementioned, the M^m-type enantiomorphic group pairs are derived from each symmetry group of rods belonging to the infinite classes p$(4k)_{2k}$ and p$(4k)_{2k}22$ $(k \in N)$. They are given by International symbols and by the symbols proposed by A.M.Zamorzaev [2], comprising their antisymmetric characteristics AC, number of AC-isomorphism class and numbers $N_m(S)$ [3].

1) p$(4k)_{2k}$, $\{c\}(c/2(4k))$, AC: $\{c\}\{c/2(4k)\}$;
   2.1, $N_1=3$, $N_2=6$;
2) p$(4k)_{2k}22$, $\{c\}(c/2(4k):2)$,
   AC: $\{c/2(4k)\}\{\{2,2c\},\{2c/2(4k),2cc/2(4k)\}\}$
   3.4, $N_1=4$, $N_2=15$, $N_3=42$.

In the case of symmetry groups of rods p$(4k)_{2k}$ and p$(4k)_{2k}22$ the transition from the initial AC to the transformed (AC^*) is realized multiplying the corresponding screw motion by c. For the recognition of M^m -type enantiomorphic group pairs we have only to compare initial and transformed antisymmetric characteristics and find the equal ones. By treating in this manner all M^m -type groups of each of two families enumerated by 1-2, we will obtain all missing enantiomorphic pairs of the M^m -type groups.

The suggested method is illustrated by example of the M^m-type groups (m=1,2) derived from the group enumerated by 1 ($p(4k)_{2k}$).

$p(4k)_{2k}$, $\{c\}(c/2(4k))$,

m=1

	AC: $\{c\}\{c/2(4k)\}$,	AC^* : $\{c\}\{c/2(4k)c\}$
1) $\{c'\}(c/2(4k))$,	$\{e_1\}\{E\}$,	$\{e_1\}\{e_1\}$;
2) $\{c\}(c/2(4k)')$,	$\{E\}\{e_1\}$,	$\{E\}\{e_1\}$;
3) $\{c'\}(c/2(4k)')$,	$\{e_1\}\{e_1\}$,	$\{e_1\}\{E\}$.

From the initial and transformed antisymmetric characteristic of the written groups we conclude that the antisymmetry groups 1 and 3 are enantiomorphic.

m=2

	AC: $\{c\}\{c/2(4k)\}$,	AC^* : $\{c\}\{c/2(4k)c\}$
1) $\{c'\}(^*c/2(4k))$,	$\{e_1\}\{e_2\}$,	$\{e_1\}\{e_1 e_2\}$;
2) $\{^*c'\}(^*c/2(4k))$,	$\{e_1 e_2\}\{e_2\}$,	$\{e_1 e_2\}\{e_1\}$;
3) $\{^*c\}(c/2(4k)')$,	$\{e_2\}\{e_1\}$,	$\{e_2\}\{e_1 e_2\}$;
4) $\{^*c\}(^*c/2(4k)')$,	$\{e_2\}\{e_1 e_2\}$,	$\{e_2\}\{e_1\}$;
5) $\{^*c'\}(c/2(4k)')$,	$\{e_1 e_2\}\{e_1\}$,	$\{e_1 e_2\}\{e_2\}$;
6) $\{c'\}(^*c/2(4k)')$,	$\{e_1\}\{e_1 e_2\}$,	$\{e_1\}\{e_2\}$.

Comparing the initial (AC) and transformed antisymmetric characteristic (AC^*) for $m=2$, we notice that 1 and 6, 2 and 5, 3 and 4, are enantiomorphic group pairs. Consequently, among the M^m-type groups derived from the symmetry group $\mathrm{p}(4k)_{2k}$ there is one enantiomorphic group pair for $m=1$ and three for $m=2$.

Moreover, directly from a generating group S it is possible to derive all different non-enantiomorphic M^m-type groups and compute their number $\overline{N}_m(S)$, avoiding the aforementioned finding procedure. Having this purpose, it is necessary to transform the initial $AC(S)$ into a new antisymmetric characteristic $\overline{AC}(S)$, establishing the equivalence between every M^m-type group derived from S, and its enantiomorph. If S is one of symmetry groups denoted by 1-2, and $AC(S)^*$ is derived by replacing in $AC(S)$ the c-screw motion by its product with the c-translation, then $\overline{AC}(S)=\{AC(S),AC(S)^*\}$.

The construction of $\overline{AC}$ is illustrated by example of the group 1 ($\mathrm{p}(4k)_{2k}\cong\{c\}(c/2(4k))$) with the antisymmetric characteristic $AC:\{c\}\{c/2(4k)\}$. By multiplying the screw motion $c/2(4k)$ with the translation c, the initial AC is transformed into the $AC^*:\{c\}\{c/2(4k)c\}$. The addition of the AC^* to the initial AC results in the $\overline{AC}:\{c/2(4k),c/2(4k)c\}$, making possible direct derivation of non-enantiomorphic M^m-type groups.

Since the $\overline{AC}$ of the group $\mathrm{p}(4k)_{2k}$ is isomorphic to the antisymmetric characteristic 2.2 [3], according to Theorem 2, we can directly conclude that $\overline{N}_1(\mathrm{p}(4k)_{2k})=N_1(2.2)=2$, $\overline{N}_2(\mathrm{p}(4k)_{2k})=N_2(2.2)=3$.

By the same method, the antisymmetric characteristic $\overline{AC}$ of the symmetry group of rods previously enumerated by 2, is obtained. The final result, including the numbers of AC-isomorphism classes (for AC and $\overline{AC}$) and the corresponding numbers

$N_m(S)$ and $\bar{N}_m(S)$ [3], is the following:

1) $p(4k)_{2k}$, $\{c\}(c/2(4k))$,

 AC: $\{c\}\{c/2(4k)\}$; $\overline{AC}$: $\{c/2(4k), c/2(4k)c\}$,

 2.1, $N_1=3$, $N_2=6$; 2.2, $\bar{N}_1=2$, $\bar{N}_2=3$;

2) $p(4k)_{2k}22$, $\{c\}(c/2(4k):2)$,

 $\overline{AC}$: $\{c/2(4k)\}\{\{2,2c\},\{2c/2(4k),2cc/2(4k)\}\}$

 AC: $\{c/2(4k)\}\{\{2,2c\},\{2c/2(4k),2cc/2(4k)\}\}$,

 3.4, $N_1=4$, $N_2=15$, $N_3=42$; 3.8, $\bar{N}_1=3$, $\bar{N}_2=9$, $\bar{N}_3=21$.

The number of M^m -type enantiomorphic group pairs derived from S is given by the formula $N_m(S)-\bar{N}_m(S)$. Consequently, among the M^m-type groups derived from the symmetry groups of rods belonging to the infinite classes 1-2 there are 2 M^1-type, 9 M^2-type and 21 M^3-type enantiomorphic group pair for every fixed k.

Having in mind the geometrical interpretation of l-multiple antisymmetry, the results obtained can be efficiently used for the investigation of the infinite classes of multi-dimensional subperiodic non-crystallographic symmetry groups of the category $G_{(l+3)(l+2)\ldots 31}$ [1,7,8,9].

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ENANTIOMORFIZAM NEKRISTALOGRAFSKIH
GRUPA PROSTE I VIŠESTRUKE ANTISIMETRIJE STOŽERA

Istraživanje enantiomorfizma kristalografskih grupa *l*-struke antisimetrije stožera [1] prošireno je na nekristalografske grupe iste kategorije.

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