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Giuseppe Di Maio*, Somashekhar Naimpally**

THE LOCALLY FINITE HYPERTOPOLOGY AND GENERALIZED UNIFORMITIES

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Abstract. Beer, Himmelberg, Prikry and Van Vleck showed that if X is a metrizable space, then the locally finite topology on 2^X is the sup of all Hausdorff metric topologies on 2^X generated by compatible metrics on X. The second author and Sharma generalized this result to normal spaces. In this paper it is shown that the locally finite topology on 2^X is the Hausdorff generalized uniform topology corresponding to a generalized uniformity on X which is Mozzocchi as well as local.

1. Introduction

In a recent paper [1] Beer et al. showed that the locally finite topology e^{τ} on the hyperspace 2^X of nonempty closed subsets of a metrizable space X is the supremum of all the Hausdorff metric topologies generated by equivalent metrics on X. Naimpally and Sharma showed that X is normal iff the locally finite topology e^{τ} is the Hausdorff uniform topology generated by the fine uniformity on X. It is obvious that if X is Tychonoff but not normal, then its fine uniformity is incapable of generating the locally finite topology e^{τ} on 2^X. A natural question then arises as to whether e^{τ} is generated by a generalized uniformity which is finer than the fine (Weil) uniformity on X. In this paper we show that there is a generalized or g-uniformity $\mathcal{V}$ on a Tychonoff space which is local (see J. Williams [7]) as well as Mozzocchi (see Mozzocchi-Gagrat-Naimpally [3] or Naimpally-Warrack [4]) and the Hausdorff g-uniformity $2^{\mathcal{V}}$ induces the locally finite topology e^{τ} on 2^X. We show that X is normal iff $\mathcal{V}$ is a uniformity and therefore all the previous results follow from our paper.

Suppose (X,τ) is a Tychonoff space and 2^X the family of all nonempty

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closed subsets of X. Let 2^{τ}, e^{τ} denote respectively the finite (Vietoris) and locally finite topology on 2^X (see Beer et al. [1]). $\mathcal{U} = \mathcal{U}(\tau)$ denotes the fine uniformity on (X,τ) and $\mathcal{U}^* = \mathcal{U}^*(\tau)$ the finest totally bounded uniformity on (X,τ). For $C, D \in 2^X$ we set

$$W_{C,D} = [X \times X \setminus (C \times D \cup D \times C)].$$

We note that if (X,δ) is an EF-proximity space, then $\{W_{C,D}: C\delta D\}$ generates the totally bounded uniformity in $\Pi(\delta)$, the family of all uniformities on X compatible with δ (see Pervin [6], p. 194). Now we set

$$\mathcal{V} = \mathcal{U} \vee \{W_{C,D}: C \cap D = \emptyset\},$$
$$\mathcal{V}^* = \mathcal{U}^* \vee \{W_{C,D}: C \cap D = \emptyset\}.$$

Suppose δ_0 denotes the finest compatible LO-uniformity on (X,τ) viz. $A\delta_0 B$ iff $\bar{A} \cap \bar{B} \neq \emptyset$. It is well known that δ_0 is EF iff (X,τ) is normal. If $\mathcal{W}$ is any g-uniformity, $\delta = d(\mathcal{W})$ is such that $A\delta B$ iff $W[A] \cap B \neq \emptyset$ for each $W \in \mathcal{W}$. If $\{U_i : i \in I\}$ is a nonempty family of subsets of X we write

$$\langle U_i : i \in I\rangle^- = \{A \in 2^X : A \cap U_i \neq \emptyset \text{ for each } i \in I\},$$
$$\langle U_i : i \in I\rangle^+ = \{A \in 2^X : A \subset \cup\{U_i : i \in I\}\},$$
$$\langle U_i : i \in I\rangle = \langle U_i : i \in I\rangle^- \cap \langle U_i : i \in I\rangle^+.$$

Furthermore, if $\mathcal{W}$ is a (generalized) uniformity on X, $|\mathcal{W}|$ denotes the topology on X induced by $\mathcal{W}$. For each $W \in \mathcal{W}$ we set

$$\mathbb{W} = \{A, B \in 2^X : A \subset W(B),\ B \subset W(A)\}.$$

Then the collection $\{\mathbb{W} : W \in \mathcal{W}\}$ is a base for a (generalized) uniformity $2^{\mathcal{W}}$ on 2^X.

2. Results

In this section we prove our main result after going through some preliminaries.

2.1. LEMMA. $\delta(\mathcal{V}^*) = \delta(\mathcal{V}) = \delta_0$.

Proof. Clearly $\delta(\mathcal{V}^*) \leq \delta(\mathcal{V}) \leq \delta_0$. $A\delta_0 B$ implies $\bar{A} \cap \bar{B} = \emptyset$. Therefore, $W_{\bar{A},\bar{B}}[A] \cap B = \emptyset$. Consequently, $\delta_0 \leq \delta(\mathcal{V}^*)$. ∎

2.2. COROLLARY. *$\mathcal{V}$ and $\mathcal{V}^*$ are Mozzocchi M-uniformities.* ∎

2.3. COROLLARY. *$\mathcal{V}$ and $\mathcal{V}^*$ are local uniformities.*

Proof. The result follows if we show that the local triangle inequality is satisfied for each $W_{C,D}$, $C, D \in 2^X$, $C \cap D = \emptyset$ and $x \in X$.

(i) If $x \in C$, then $W_{C,D}[x] = X \setminus D$.

(ii) If $x \in D$, then $W_{C,D}[x] = X \setminus C$.

(iii) If $x \notin C \cup D$, then $W_{C,D}[x] = X$.

Case (iii) is trivial and (i) and (ii) are similar. Thus we consider (i). Since $x \notin D$ there is a continuous function $f: X \to [0,1]$ such that $f(x) = 0$, $f(D) = 1$. Set $V = \{(x,y): |f(x) - f(y)| < 1/4\} \in \mathcal{U}^*$. Then $V^2[x] \subset W_{C,D}[x] = X \setminus D$. ■

2.4. THEOREM. *The following are equivalent:*

(a) (X,τ) *is normal.*

(b) $\mathcal{V} = \mathcal{U}$.

(c) $\mathcal{V}^* = \mathcal{U}^*$.

Proof. If $\mathcal{V} = \mathcal{U}$ or $\mathcal{V}^* = \mathcal{U}^*$, then δ_0 is EF and so (X,τ) is normal. If (X,τ) is normal, δ_0 is EF and $\{W_{C,D}: C\delta_0 D\} = \{W_{C,D}: C \cap D = \emptyset\}$ is a base for $\mathcal{U}^*$. Hence $\mathcal{V}^* = \mathcal{U}^*$. Since $\mathcal{U}^* \subset \mathcal{U}$, $\mathcal{V} = \mathcal{U}$. ■

2.5. THEOREM. $|2^{\mathcal{V}^*}| = 2^\tau$.

Proof. Suppose $\{U_i: 1 \le i \le n\} \subset \tau$, $A \in 2^X$ and $A \in \langle U_i \rangle$, i.e. for each $i \le n$, there is an $a_i \in A \cap U_i$ and $A \subset U = \cup\{U_i: 1 \le i \le n\}$. Then there is a $V \in \mathcal{U}^*$ such that $V[a_i] \subset U_i$ and $W_{A,X-U}[A] \subset U_i$. Set $V' = V \cap W_{A,X-U} \in \mathcal{V}^*$. It is easy to see that $A \in \mathbb{V}'[A] \subset \langle U_i \rangle$, i.e. $2^\tau \subset |2^{\mathcal{V}^*}|$.

Clearly $|2^{\mathcal{U}^*}| \subset 2^\tau$ by Michael [2;Th. 3.2]. Consider

$$\mathbb{W}_{C,D}[A] = \{E \in 2^X: E \subset W_{C,D}[A] \text{ and } A \subset W_{C,D}[E]\}.$$

(i) $A \cap D = \emptyset$ implies $A \subset X \setminus D$, i.e. $A \in \langle X \setminus D\rangle^+ \subset \mathbb{W}_{C,D}[A]$.

(ii) $A \cap D \neq \emptyset$ implies $A \in \langle X \setminus C\rangle^- \subset \mathbb{W}_{C,D}[A]$.

So, $|2^{\mathcal{V}^*}| \subset 2^\tau$. ■

Now we prove the main result of our paper.

2.6. THEOREM. $|2^{\mathcal{V}}| = e^\tau$.

Proof. (a) $|2^{\mathcal{V}}| = e^\tau$. If $V \in \mathcal{U}$, then Naimpally-Sharma [4;Th. 2.1] have shown that $\mathbb{V}[A]$ is open in e^τ for $A \in 2^X$. The case $V = W_{C,D}$ can be treated as in the previous theorem. So $|2^{\mathcal{V}}| \subset e^\tau$.

(b) $e^\tau \subset |2^{\mathcal{V}}|$. If $A \in \langle U_i\rangle^-$, where $\{U_i: i \in I\}$ is a locally finite family of open sets in (X,τ), then the proof of Naimpally-Sharma [4;Th. 2.2] shows that $\langle U_i\rangle^- \in |2^{\mathcal{V}}|$. Next suppose that $A \subset W \in \tau$. Then $A \in \mathbb{W}_{A,X-W}[A] \subset W^+ \in 2^\tau$. ■

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Giuseppe Di Maio, Somashekhar Naimpally

LOKALNO KONAČNA HIPERTOPOLOGIJA I GENERALISANE UNIFORMNOSTI

Neka je 2^X skup svih nepraznih zatvorenih podskupova datog topološkog prostora X. U radu se pokazuje da na svakom potpuno regularnom prostoru X postoji generalisana uniformnost koja je lokalna i Mozzocchijeva i takva da njom generisana uopštena Hausdorffova uniformnost na 2^X indukuje lokalno konačnu topologiju na 2^X.

Giuseppe Di Maio
Dipartimento di Matematica
Via Mezzocanone 8
80134 Napoli
Italy

Somashekhar Naimpally
Department of Mathematics
Carleton University
Ottawa
K1S 5B6 Canada