Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 5(1991), 109-112

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ABSTRACT MEASURE OF FARNESS AND WIJSMAN CONVERGENCE

(Received 06.03.1991.)

Abstract. Bella, Di Maio and Naimpally introduced in [2] the notion of abstract measure of farness for uniform spaces and generalized several results of Beer, Lechicki, Levi and Naimpally concerning with the Vietoris hypertopology and proximal hypertopology [1]. In this short note the programme is continued by attacking the Wijsman topology.

1. Introduction

In this paper (X, τ_0) denotes a Tychonoff space, 2^X the power set of X and CL(X) the family of all nonempty closed subsets of X. In case X is metrizable with metric d, $\delta = \delta(d)$ denotes the EF-proximity induced by d on X, i.e. $A\delta B$ iff $d(A,B) = 0$. Recently Beer et al [1] showed that if X is a metrizable space with $\mathcal{D}$ the set of all admissible metrics, then the following results hold:

1.1. ([1], Theorem 3.1) *The Vietoris topology on CL(X) is the supremum of all Wijsman topologies corresponding to each* $d \in \mathcal{D}$. ■

1.2. ([1], Theorem 3.7) *For each* $d \in \mathcal{D}$, *the proximal hypertopology with respect to d is the supremum of all Wijsman topologies corresponding to* $\{\rho \in \mathcal{D}: \delta(\rho) = \delta(d)\}$. ■

In this article we build on the terminology and notation of [2] which we review for the sake of completeness.

Let $\mathcal{B}$ denote a compatible uniformity base consisting of open symmetric members of $X \times X$. We assume that members of $\mathcal{B}$ are generated by a gage $\mathcal{P}$ of pseudometrics, i.e. each member U of $\mathcal{B}$ is of the form $U = U^d(\varepsilon) = \{(x,y): d(x,y) < \varepsilon\}$, where $d \in \mathcal{P}$, $\varepsilon > 0$. $\mathcal{U}_0$ denotes the fine uniformity on X, $\mathcal{U}^*$

AMS Subject Classification (1991): 54B20, 54A10, 54E05, 54E15
(*) Supported by C.N.R. (G.N.S.A.G.A.) and M.U.R.S.T. through "Fondi 40%"
(**) Supported by a visiting professorship under C.N.R.(Italy) in July 1990

denotes the Čech uniformity which is the finest totally bounded uniformity on X. let $\delta = \delta(\mathcal{B})$ be the EF-proximity induced on X by $\mathcal{B}$. We write A<<B for $A\bar{\delta}(X\setminus B)$ and $A\delta_0 B$ for $\mathrm{Cl}A \cap \mathrm{Cl}B \neq \emptyset$. Each $d \in \mathcal{P}$ gives rise to a Hausdorff pseudometric H_d on CL(X). The Hausdorff uniformity $\mathcal{H}(\mathcal{B})$ on CL(X) is generated by $\{\mathcal{H}_d : d \in \mathcal{P}\}$. $\tau_{\mathcal{H}(\mathcal{B})} = \tau_{\mathcal{H}}$ denotes the topology on CL(X) induced by $\mathcal{H}(\mathcal{B})$. For each $G \in \tau_0$, we denote by $G^{++} = \{F \in CL(X): F<<G\}$. So the upper proximal topology τ_{δ^+} is generated by $\{G^{++}: G \in \tau_0\}$. And the proximal topology τ_δ is the sup of τ_{V-} and τ_{δ^+}.

The following are known (see [1], [3], [4]):

1.3. THEOREM. *(a)* $\tau_\delta \subset \tau_{\mathcal{H}}$, $\tau_\delta \subset \tau_V$.

(b) $\tau_\delta = \tau_V$ *if and only if* $\delta = \delta_0$ *and in this case* (X,τ_0) *is normal.*

(c) $\tau_\delta = \tau_{\mathcal{H}}$ *if and aonly if* $\mathcal{B}$ *is totally bounded.* ■

Let $\mathcal{B}$ be an open symmetric uniformity base on X and let $\mathcal{C} = 2^{\mathcal{B}}$ be partially ordered by reverse inclusion. The abstract measure of farness φ is a function on $2^X \times 2^X \to \mathcal{C}$ given by

$$\varphi(A,B) = \begin{cases} \{U \in \mathcal{B}: U \cap (A\times B) \neq \emptyset\}, & \text{if } A \neq \emptyset \text{ and } B \neq \emptyset \\ \emptyset, & \text{if A or B is empty.} \end{cases}$$

We say that $U' \in \mathcal{U}$ is composably contained in $U \in \mathcal{U}$ iff there exists an $U'' \in \mathcal{U}$ such that $U' \circ U'' \subset U$. Let φ be the abstract measure of farness and τ a topology on CL(X). We say that φ is USC iff for $U \in \varphi(A,B)$ there exist τ-neighborhoods N_A, N_B of A, B, respectively, such that for each $C \in N_A$, $D \in N_B$, $U' \notin \varphi(C,D)$. For $A \in CL(X)$, we set $\varphi_A : CL(X) \to \mathcal{C}$ by

$$\varphi_A(F) = \varphi(A,F)$$

and use the terms USC, LSC etc, analogously. We say that φ (φ_A) is continuous iff it is USC and LSC.

The following results have been proved in [2]:

1.4. LEMMA. *Let* $\mathcal{B}$ *be an open symmetric uniformity base compatible with* τ_0. *Let* τ *be a topology on CL(X) and let* $\varphi = \varphi_{\mathcal{B}}$. *Then the following are equivalent:*

(a) $\tau_{V-} \subset \tau$.

(b) $\varphi : (CL(X),\tau) \times (CL(X),\tau) \to \mathcal{C}$ *is USC.*

(c) For each $A \in CL(X)$, $\varphi_A : (CL(X),\tau) \to \mathcal{C}$ *is USC.* ■

1.5. LEMMA. *With the conditions as in Lemma 1.4, the following are equivalent:*

(a) $\tau_{\delta^+} \subset \tau$.

(b) $\varphi : (CL(X),\tau) \times (CL(X),\tau) \to \mathcal{C}$ *is LSC.*

(c) For each $A \in CL(X)$, $\varphi_A : (CL(X),\tau) \to \mathcal{C}$ *is LSC.* ■

1.6. THEOREM. *Under the conditions of Lemmas 1.4 and 1.5 the following are equivalent:*

(a) $\tau_\delta \subset \tau$.

(b) $\varphi:(CL(X),\tau) \times (CL(X),\tau) \to \mathscr{C}$ *is continuous.*

(c) For each $A \in CL(X)$, $\varphi_A:(CL(X),\tau) \to \mathscr{C}$ *is continuous.* ■

2. The Wijsman hypertopology

In this section we study the Wijsman hypertopology [6]. The lower Wijsman topology $\tau_{W^-} = \tau_{V^-}$ and the upper Wijsman topololy τ_{W^+} is defined to be the weakest topology on CL(X) such that the function $\varphi_x:CL(X) \to \mathscr{C}$ defined by

$$\varphi_x(F) = \varphi(x,F) \quad \text{(where we write x for } \{x\})$$

is LSC for each $x \in X$.

2.1. DEFINITION. *The Wijsman topology* τ_W *is the sup of* τ_{W^-} *and* τ_{W^+}, *namely* $\tau_W = \tau_{W^-} \vee \tau_{W^+}$. ■

2.2. REMARK. τ_W is the weakest topology on CL(X) such that for each x in X, φ_x is continuous. As a consequence of Lemma 1.6 we have:

$$\tau_W(\mathscr{B}) \subset \tau_\delta(\mathscr{B}). \quad ■$$

2.3. THEOREM. *If* $\mathscr{B}$ *is totally bounded, then* $\tau_W(\mathscr{B}) = \tau_\delta(\mathscr{B})$.

Proof. By Lemma 1.5 it is sufficient to prove that if $\varphi_x:(CL(X),\tau) \to \mathscr{C}$ is LSC for each $x \in X$, then $\varphi_A:(CL(X),\tau) \to \mathscr{C}$ is LSC for each $A \in CL(X)$. If $\varphi_A(B) = 0$ for $B \in CL(X)$ we are done. If not suppose $U \notin \varphi_A(B) = \varphi(A,B)$. Let U' be any entourage composably contained in U and $V \in \mathscr{B}$ such that $U' \circ V^3 \subset U$. Since $\mathscr{B}$ is totally bounded, there is some positive integer n such that $A \subset \cup_{i=1}^n V(a_i)$, $a_i \in A$. Since φ_{a_i} is LSC at B, there exists $N_i \in \tau$ such that $B \in N_i$ and $F \in N_i$ implies $U' \circ V^2 \notin \varphi_{a_i}(F)$. We claim that if $F \in N = \cap_{i=1}^n N_i$, a τ-neighborhood of B, then $U' \notin \varphi_A(F)$. $F \in N$ implies $F \in N_i$ for each $i \le n$ and consequently $F \cap U' \circ V[V(a_i)] = \emptyset$. Hence, $F \cap U'[A] = \emptyset$ and so φ_A is LSC. ■

2.4. COROLLARY. $\tau_W(\mathscr{B}) \subset \tau_\delta(\mathscr{B}) \subset \tau_{\mathscr{H}}(\mathscr{B})$. *All these topologies are equal if and only if* $\mathscr{B}$ *is totally bounded.* ■

A metrizable space need not have a compatible totally bounded metric unless it is separable and hence second countable. On the other hand, every Tychonoff space has a compatible finest totally bounded uniformity $\mathscr{U}^*$, the Čech uniformity. So if (X,τ_0) is normal, then $\delta_0 = \delta(\mathscr{U}^*)$ and $\tau_W(\mathscr{U}^*) = \tau_\delta(\mathscr{U}^*) = \tau_{\delta_0} = \tau_{\mathscr{H}}(\mathscr{U}^*)$ (see Theorem 1.3). So we have one of our main results:

2.5. THEOREM. *Let (X,τ_0) be normal. Then τ_V is the weakest topology on $CL(X)$ such that φ_x is continuous for each $x \in X$, where φ_x is defined with respect to any uniformity base $\mathcal{B}$ for $\mathcal{U}^*$.* ■

If (X,τ_0) is second countable, then it is metrizable and $\mathcal{U}^*$ is generated by $\mathcal{D}^*$, the set of all admissible totally bounded metrics on X. So we have

2.6. THEOREM. (cf. 1.1) *If X is a second countable metrizable space, then τ_V is the weakest topology on $CL(X)$ such that for each $x \in X$, $d \in \mathcal{D}^*$, $d:CL(X) \to \mathbb{R}$, defined by $d(x,A) = inf\{d(x,a):a \in A\}$, is continuous.* ■

2.7. THEOREM. (cf. 1.2) *If δ is any compatible EF-proximity on (X,τ_0), then $\tau_\delta = sup\{\tau_W(\mathcal{B}):\delta(\mathcal{B}) = \delta\}$.*

Proof. $\tau_\delta(\mathcal{B}) = \tau_\delta(\mathcal{B}^\omega) = \tau_W(\mathcal{B}^\omega)$, where $\mathcal{B}^\omega$ is the totally bounded member in $\Pi(\delta)$, the proximity class of δ (see [5], p. 71). ■

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APSTRAKTNA MERA UDALJENOSTI I WIJSMANOVA KONVERGENCIJA

Apstraktna mera udaljenosti (podskupova datog topološkog prostora) koristi se za izučavanje topologije Wijsmana na skupu zatvorenih podskupova topološkog prostora.

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