Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 4(1990), 95-97

A. A. Borubaev

ON COMPLETENESS AND COMPLETIONS OF UNIFORMLY CONTINUOUS MAPPINGS

(Received 03.05.1990.)

Abstract. We introduce the notion of complete uniformly continuous mappings. A completion of a uniformly continuous mapping is defined.

A uniformly continuous mapping $f:(X,U) \to (Y,V)$ of a uniform space (X,U) onto a uniform space (Y,V) is called **complete** if for every Cauchy filter F in (X,U) such that $f(F)$ is convergent in (Y,V), it follows that F is convergent in (X,U).

Let us consider the following diagram

$$(1)\qquad \begin{array}{ccc} (X,U) & \xrightarrow{\ i_X\ } & (\tilde{X},\tilde{U}) \\ \Big\Downarrow f & & \Big\Downarrow \tilde{f} \\ (Y,V) & \xrightarrow{\ i_Y\ } & (\tilde{Y},\tilde{V}) \end{array},$$

where $(\tilde{X},\tilde{U})$ and $(\tilde{Y},\tilde{V})$ are completions of spaces (X,U) and (Y,V), respectively, i_X and i_Y canonical embeddings of (X,U) and (Y,V) in $(\tilde{X},\tilde{U})$ and $(\tilde{Y},\tilde{V})$ and $\tilde{f}$ a unique uniformly continuous extension of the mapping f. This diagram is commutative and it is a pull-back in the category **Unif**.

THEOREM 1. *Let $f:(X,U) \to (Y,V)$ be a uniformly continuous mapping. Then the following conditions are equivalent:*

(1) The mapping f is complete;

(2) The diagram (1) is a pull-back in the category **Unif**;

(3) $\tilde{f}(\tilde{X} \setminus X) \subset \tilde{Y} \setminus Y$.

Proof. (1) $\Rightarrow$ (2). Let f be complete and let $\varphi:(Z,W) \to (\tilde{X},\tilde{U})$ and $\psi:(Z,W) \to (\tilde{Y},\tilde{V})$ be uniformly continuous mappings such that $\tilde{f}\cdot\varphi = i_Y\cdot\psi$. Let us show that the diagram (1) is a pull-back in the category **Unif**, i.e. there is a (unique) uniformly continuous mapping $h:(Z,W) \to (X,U)$

AMS Subject Classification (1980): 54E15, 54E52 such that $i_X \cdot h = \varphi$ and $f \cdot h = \psi$. Let z be an arbitrary element in Z and $\mathcal{B}$ a filter of neighbourhoods of the point $\varphi(z)$ in $(\tilde{X}, \tilde{U})$. Then $F = i_X^{-1}(\mathcal{B})$ is a Cauchy filter in (X, U). As $\tilde{f}(\varphi(z)) = i_Y(\psi(z))$, it is not difficult to see that f(F) converges to the point $\varphi(z)$ in (Y, V). Then, according to the definition of a complete mapping, the filter F converges to a point $x \in X$. Now putting h(z) = x we have defined the mapping $h: Z \to X$ which is uniformly continuous and satisfies the conditions $i_X \cdot h = \varphi$ and $f \cdot h = \psi$.

(2) ⇒ (3). Let $y \in Y$ and $y \notin \tilde{Y} \setminus Y$. We will show that $y \notin \tilde{f}(\tilde{X} \setminus X)$. Consider the one-point uniform space Z = {y} with the trivial uniformity W. Let $p \in \tilde{X}$ be an element such that $f(p) = i_Y(y)$. Putting $\psi(y) = y$ and $\varphi(y) = p$, we have defined mappings $\varphi: Z \to \tilde{X}$ and $\psi: Z \to Y$ for which, by the construction, we have $\tilde{f} \cdot \varphi = i_Y \cdot \psi$. Then, according to condition (2), one has $i_Y \cdot h = \varphi$ and $f \cdot h = \psi$. Hence, $p \in X$ and thus $\tilde{f}^{-1}(y) \in X$. This means that $y \notin \tilde{f}(\tilde{X} \setminus X)$ and the inclusion (3) is proved.

(3) ⇒ (1). Let F be a Cauchy filter in (X, U) such that f(F) converges to a point y in (Y, V). Then F is a base of some Cauchy filter in the complete space $(\tilde{X}, \tilde{U})$ and it converges to a point x in $(\tilde{X}, \tilde{U})$. Then $\tilde{f}(x) = y$. From $\tilde{f}(\tilde{X} \setminus X) \subset \tilde{Y} \setminus Y$ it follows that $x \in X$. Therefore, F converges (in $(\tilde{X}, \tilde{U})$) to the point $x \in X$, i.e. f is a complete mapping. The theorem is proved.

Let $f: (X, U) \to (Y, V)$ be a uniformly continuous mapping. A uniformly continuous mapping $f^*: (X^*, U^*) \to (Y, V)$ is called a **completion** of f if the following conditions are satisfied:

1) (X, U) is a dense uniform subspace of (X^*, U^*);
2) The mapping f^* is complete;
3) $f = f^* | X$.

THEOREM 2. *Every uniformly continuous mapping has only one (up to a uniform isomorphism) completion.*

Proof. Let $f: (X, U) \to (Y, V)$ be uniformly continuous. Let $\tilde{f}: (\tilde{X}, \tilde{U}) \to (\tilde{Y}, \tilde{V})$ be the unique uniformly continuous extension of the mapping f to the completions $\tilde{X}$ and $\tilde{Y}$ of X and Y, respectively. Put $X^* = \tilde{f}^{-1}(Y)$. Let U^* be the uniformity on X^* induced by the uniformity $\tilde{U}$ and $f^* = \tilde{f} | X^*$. It is easy to see that $f^*: (X^*, U^*) \to (Y, V)$ is a completion of f.

A uniformly continuous mapping $f: (X, U) \to (Y, V)$ is called **uniformly perfect** (see [1]) if it is uniformly continuous precompact and perfect (in the topological sense).

THEOREM 3. *A mapping $f:(X,U) \to (Y,V)$ is uniformly perfect iff it is complete and precompact.*

Proof. Clearly, if f is uniformly perfect, then f is complete. Let f be a complete precompact mapping. We are going to show that f is perfect. (i) f is a compact mapping: Take any $y \in Y$. Let F_y be an arbitrary Cauchy filter in $(f^{-1}(y),U_y)$. Then $f(F_y)$ converges to the point y and because of completeness of f, F_y converges in $(f^{-1}(y),U_y)$. Hence, $(f^{-1}(y),U_y)$ is a complete subspace of (X,U). On the other hand, from precompactness of f it follows that $(f^{-1}(y),U_y)$ is precompact in (X,U). So, $(f^{-1}(y),U_y)$ is compact. (ii) f is closed: Let $y \in Y$ and let O be an open set containing $f^{-1}(y)$. We will show that there exists an open set G such that $y \in G$ and $f^{-1}(G) \subset O$. Suppose, on the contrary, that for every $\beta \in V$, $f^{-1}(\beta(y)) \cap (X \setminus O) \neq \emptyset$. Let F be a ultrafilter in X containing the centered system $\{f^{-1}(\beta(y)) \cap (X \setminus O):\beta \in V\}$. Then $\gamma \cap F \neq \emptyset$ for every finite covering $\gamma \in U$. By the construction, $f^{-1}(\beta) \cap F \neq \emptyset$ for every $\beta \in V$. The precompactness of f implies that $\alpha \cap F \neq \emptyset$ for every $\alpha \in U$, i.e. F is a Cauchy filter in (X,U). It is clear that $f(F)$ converges to the point $y \in Y$. Then, by the definition of a complete mapping, F converges to some point $x \in X$. Thus $\cap\{f^{-1}(\beta(y)) \cap (X \setminus O):\beta \in V\} \neq \emptyset$. However, it is easy to see that we came to a contradiction. Hence, f is a closed mapping.

THEOREM 4. *A uniformly continuous mapping f is precompact iff its completion is uniformly perfect.*

REFERENCES

[1] A.A. BORUBAEV, *Uniformly perfect mappings. Absolutes of uniform spaces,* Comp. Rend. Acad. Bulg. Sci. 42(1989), 19-21 (in Russian).

A.A. Borubaev

O KOMPLETNOSTI I KOMPLETIRANJU UNIFORMNO NEPREKIDNIH PRESLIKAVANJA

Definiše se kompletno i uniformno savršeno uniformno neprekidno preslikavanje i uvodi pojam kompletiranja uniformno neprekidnih preslikavanja. Proučene su neke veze medju ovim preslikavanjima.

University of Frunze
720024 Frunze, USSR