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A CHARACTERIZATION OF ω_μ-METRIZABLE SPACES

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Abstract. We define the k-developability degree of a topological space and prove that a regular space X is ω_μ-metrizable iff it is ω_μ-additive and its k-developability degree is ω_μ.

In [3], R.E. Hodel defined the **metrizability degree** m(X) of a topological space X to be the smallest infinite cardinal τ such that there is a base for X consisting of τ discrete collections. Hodel also gave several characterizations of m(X) and extended some metrization theorems to higher cardinality.

In [4] P. Nyikos and H.-C. Reichel introduced the **developability degree** of a regular space and showed its importance in metrization theory.

DEFINITION 1.([4]) The developability degree of a regular space X, denoted by dv(X), is the smallest infinite cardinal τ such that there exists a collection $\{\mathcal{U}_\alpha : \alpha \in \tau\}$ of open covers of X such that the family $\{St(x,\mathcal{U}_\alpha) : \alpha \in \tau\}$ is a local base at any point $x \in X$.

(As usual, for $A \subset X$, $St(A,\mathcal{U}_\alpha) = \cup\{U : U \in \mathcal{U}_\alpha,\ A \cap U \neq \emptyset\}$. For $x \in X$ we write $St(x,\mathcal{U}_\alpha)$ instead of $St(\{x\},\mathcal{U}_\alpha)$. $St^2(x,\mathcal{U}_\alpha) = St(St(x,\mathcal{U}_\alpha),\mathcal{U}_\alpha)$.)

It is known that the following metrization theorem is true: a Hausdorff space X is metrizable iff it has a contable collection $\{\mathcal{U}_n : n \in N\}$ of open covers such that for every compact set $C \subset X$ and every neighborhood U of C there is some $i \in N$ for which $St(C,\mathcal{U}_i) \subset U$ (see [1;T.1.1] and [2;3.4.E]).

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Here we define the k-developability degree of a regular space and extend this result to higher cardinality.

DEFINITION 2. *The* **k-developability degree** *$kdv(X)$ of a regular space X is the smallest infinite cardinal τ such that there exists a family $\{\mathcal{U}_\alpha : \alpha \in \tau\}$ of open covers of X with the property that for every compact set $C \subset X$ and every its neighborhood V there is a $\mathcal{U}_\alpha$ with $St(C, \mathcal{U}_\alpha) \subset V$.*

Clearly, $dv(X) \le kdv(X)$ for every regular space X. There is a space Y such that $dv(Y) < kdv(Y)$ (see [1]).

PROPOSITION 1. *For any regular space X, $kdv(X) \le m(X)$.*

Proof. Let $m(X) = \tau$. We use the characterization "(MM)" of $m(X)$ proved by R.E. Hodel in [3]:

(MM) X has a collection $\{\mathcal{U}_\alpha : \alpha \in \tau\}$ of open covers such that for every point $x \in X$, $\{St^2(x, \mathcal{U}_\alpha) : \alpha \in \tau\}$ is a local base at x.

Denote by Λ the set of all finite subsets $\lambda = (\alpha_1, \dots, \alpha_k)$ of τ. Of course, $|\Lambda| \le \tau$. For $\lambda = (\alpha_1, \dots, \alpha_k) \in \Lambda$ put

$$\mathcal{V}_\lambda = \wedge\{\mathcal{U}_{\alpha_i} : 1 \le i \le k\} \equiv \{U_{\alpha_1} \cap \dots \cap U_{\alpha_k} : U_{\alpha_i} \in \mathcal{U}_{\alpha_i},\ 1 \le i \le k\}.$$

We shall prove that the collection $\{\mathcal{V}_\lambda : \lambda \in \Lambda\}$ witnesses $kdv(X) \le \tau$. Let C be any compact subset of X and W any neighborhood of C. For every $x \in C$ there exists some $\alpha(x) \in \tau$ such that $x \in St(x, \mathcal{U}_{\alpha(x)}) \subset St^2(x, \mathcal{U}_{\alpha(x)}) \subset W$. The family $\{St(x, \mathcal{U}_{\alpha(x)}) : x \in C\}$ is an open cover of C so that one can find a finite set $\{x_1, \dots, x_k\} \subset C$ such that $C \subset \cup\{St(x_i, \mathcal{U}_{\alpha_i}) : 1 \le i \le k\}$, where $\mathcal{U}_{\alpha_i} = \mathcal{U}_{\alpha(x_i)}$. Let $\lambda = (\alpha_1, \dots, \alpha_k) \in \Lambda$. We are going to prove $St(C, \mathcal{V}_\lambda) \subset W$ Let $p \in St(C, \mathcal{V}_\lambda)$. There exists a $V \in \mathcal{V}_\lambda$ such that $p \in V$ and $V \cap C \ne \emptyset$. Let $q \in V \cap C$. Then $p \in St(q, \mathcal{V}_\lambda)$. But $q \in St(x_j, \mathcal{U}_{\alpha_j})$ for some $j \le k$ so that we have:

$$p \in St(q, \mathcal{V}_\lambda) \subset St(q, \mathcal{U}_{\alpha_j}) \subset St(St(x_j, \mathcal{U}_{\alpha_j}), \mathcal{U}_{\alpha_j}) = St^2(x_j, \mathcal{U}_{\alpha_j}) \subset W.$$

This means $St(C, \mathcal{V}_\lambda) \subset W$ and the proposition is proved.

In [4], it was shown that $dv(X) = m(X)$ for every LOTS X. So we have

COROLLARY 1. *For any LOTS X we have $dv(X) = kdv(X) = m(X)$.*

Recall that a space X is **τ-additive**, τ a cardinal, if the intersection of less then τ open sets is open.

PROPOSITION 2. *If X is a regular τ-additive space with $kdv(X) = \tau$, then $m(X) \le \tau$.*

Proof. Let us suppose that a collection $\{\mathcal{U}_\alpha : \alpha \in \tau\}$ satisfies the conditions of the definition of kdv(X). Using τ-additivity of X we can assume that for every $\alpha, \beta \in \tau$, $\alpha > \beta$ implies $\mathcal{U}_\alpha$ is a refinement of $\mathcal{U}_\beta$. We shall prove that this collection satisfies condition (AS) in Hodel's paper [3] which is a characterization of m(X):

(AS) X has a collection $\{\mathcal{U}_\alpha : \alpha \in \tau\}$ of open covers such that for every point $x \in X$ and every neighborhood W of x there is a neighborhood V of X and $\beta \in \tau$ such that $St(V, \mathcal{U}_\beta) \subset W$.

Suppose on the contrary that this is not true. Then there exists a point $x \in X$ and its neighborhood W such that for every neighborhood V of x and every $\alpha \in \tau$ one has $St(V, \mathcal{U}_\alpha) \cap (X \setminus W) \neq \emptyset$. Therefore, for every $\alpha \in \tau$ there is $x_\alpha \in St(x, \mathcal{U}_\alpha)$ for which $St(x_\alpha, \mathcal{U}_\alpha) \cap (X \setminus W) \neq \emptyset$. Using the fact that $\{St(x, \mathcal{U}_\alpha) : \alpha \in \tau\}$ is a local base at x and for every $\alpha > \beta$ $St(x, \mathcal{U}_\alpha) \subset St(x, \mathcal{U}_\beta)$ one can easily verify that the τ-sequence $(x_\alpha : \alpha \in \tau)$ converges to x. So, $C = \{x_\alpha : \alpha \in \tau\} \cup \{x\}$ is a compact subset of X. Now for every $\alpha \in \tau$ we have: $St(C, \mathcal{U}_\alpha) \cap (X \setminus W) \supset St(x_\alpha, \mathcal{U}_\alpha) \cap (X \setminus W) \neq \emptyset$ which is a contradiction. This completes the proof.

From Propositions 1 and 2 we obtain

THEOREM 1. *If X is a regular τ-additive space, then $m(X) = \tau$ if and only if $kdv(X) = \tau$.*

Recall that a Tychonoff space X is called **ω_μ-metrizable** if its topology can be induced by a uniformity having a linearly ordered base of cofinality ω_μ.

Using a result of Wang Shu-Tang (see [3]) that a regular space X is ω_μ-metrizable iff it is ω_μ-additive and $m(X) = \omega_\mu$ we have

COROLLARY 2. *A regular space X is ω_μ-metrizable if and only if it is ω_μ-additive and $kdv(X) = \omega_\mu$.*

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JEDNA KARAKTERIZACIJA ω_μ-METRIZABILNIH PROSTORA

Uvodi se kardinalna funkcija k-Moore-ov stepen kdv(X) datog regularnog prostora X i koristi se radi karakterizacije ω_μ-metrizabilnih prostora: regularan prostor X je ω_μ-metrizabilan ako i samo ako je ω_μ-aditivan i kdv(X) = ω_μ.

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